Tuesday, April 20, 2010

Handlebody of genus \( g \)

Let \( U = \)

Let \( \Sigma = \partial U \)

Heegaard decomposition: \( Y = U_1 \cup U_2 \)

\[ U_1 = U_2 = \text{a handlebody of genus } g \]

Ex: \( S^3 = \)

\( S^2 \)

\( U \)

\( T^2 \)

\( U \)

\( U \)

Genus \( g \) decomposition of \( S^3 \)

Ex: \( L(p, q) = \)

\[ m \rightarrow \mathbb{P}L + qM \]

1. Glue in neighborhood of meridional disk using \( m \rightarrow \mathbb{P}L + qM \).
2. Glue in remaining 3-ball.

Note: Heegaard decomposition of \( L(p, q) \) depends only on curves \( m, \mathbb{P}L + qM \).

\( L(p, q) = \)
Theorem 2.1 (Singer 1933): Let $Y$ be an oriented closed 3-manifold.

Then $Y$ admits a Heegaard decomposition.

**Proof:** Start with a triangulation of $Y$.

Same genus

0-handles $U$

1-handles

= handlebody

3-handles $V$

2-handles

= handlebody

dual

3-handles

0-handles:

$D^3 \times S^0$

Add to each vertex

for each edge

Glue on $D^2 \times D^1$

1-handles

2-handles

= handlebody

504 x $D^3$

for each face

Glue $D^1 \times D^2$

for each tetrahedron

Stabilization of $Y = U_1 \cup U_2$

$U_i$ are genus $g$ handlebodies.

$\Sigma'$

Drill out

unknotted arc = A

$(U_1 \setminus \text{nbhd}(A)) \cup (U_2 \cup \text{nbhd}(A))$

$\partial \Sigma'$

$q(\Sigma') = q(\Sigma) + 1$

**Note:** Removing a handle results in destabilization.

Let $Y = U_1 \cup U_2 = \tilde{U}_1 \cup \tilde{U}_2$

$\Sigma \leftrightarrow \text{genus } \tilde{g} \leftrightarrow \text{genus } \tilde{g}$

Then for $k$ large enough, the $k-g$-fold stabilization of $U_1 \cup \Sigma \cup U_2$ is diffeo with $(k-g)$-fold stable of $U_1 \cup U_2$. 
**Note:** Stabilization/destabilization is like Reidemeister moves for manifolds.

If \( f : 3\text{-manifolds} \rightarrow X \) does not change under stab/destab, then it is an invariant of 3-manifolds.

**Def:** A set of attaching circles \( \{Y_1, \ldots, Y_g\} \) for \( U \), a genus \( g \) handlebody, is a collection of closed embedded curves \( \partial U = \Sigma_g \).

1. \( Y_i \cap Y_j = \emptyset \text{ } \forall i \neq j \)
2. \( \Sigma_g - Y_1 - \cdots - Y_g \) is connected, i.e. \( \{1, \ldots, g\} \) are li. in \( H_1(\Sigma, \mathbb{Z}) \)
3. \( Y_i \) bound disjoint embedded disks in \( U \).

Let \((\Sigma, U_1, U_2)\) be a genus \( g \) H.D. for \( Y \). A compatible Heegaard diagram is given by \((\Sigma, \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g)\) attaching circles for \( U_1 \) for \( U_2 \).