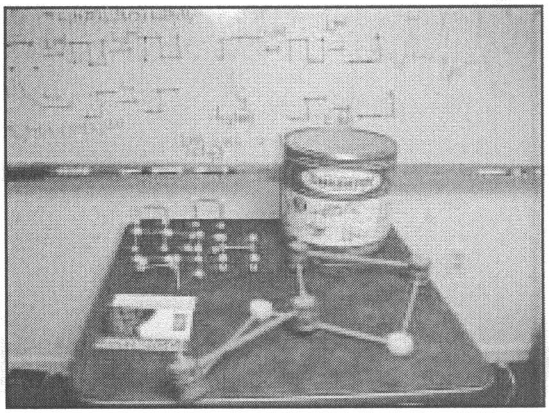

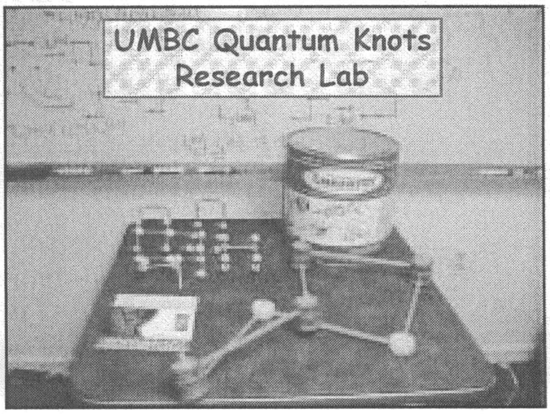

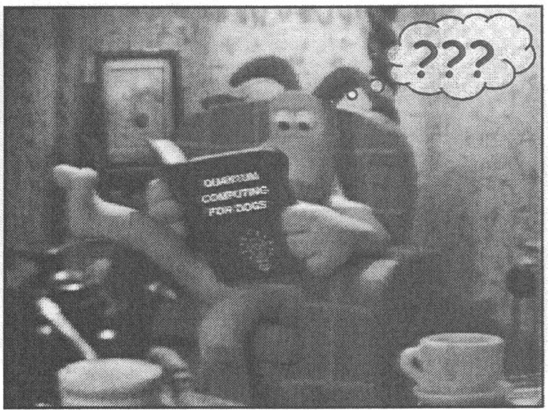

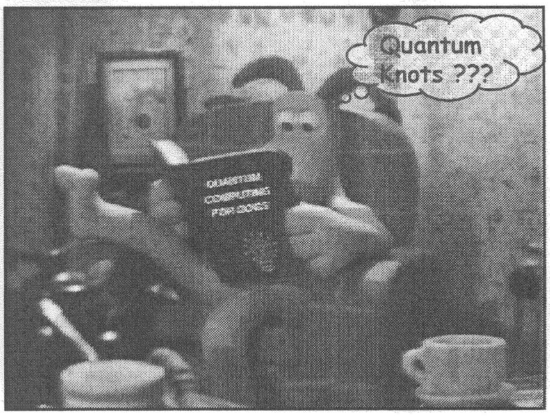

Quantum Knots & Lattices

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L-O-O-P

How to Design a Quantum System that Does Rope Tricks

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L-O-O-P

or, How
Wiggle, Wag, & Tug
Go Quantum

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L-O-O-P

Throughout this talk:

"Knot" means either a knot or a link

This talk is based on the paper:

Lomonaco and Kauffman, Quantum Knots and Lattices, to appear soon on quant-ph

This talk was motivated by:

Lomonaco and Kauffman, Quantum Knots and Mosaics, Journal of Quantum Information Processing, vol. 7, Nos. 2-3, (2008), 85-115. An earlier version can be found at: http://arxiv.org/abs/0805.0339

This talk was also motivated by:

Kauffman and Lomonaco, Quantum Knots, SPIE Proc. on Quantum Information & Computation II, (2004), 5436-30, 268-284. http://xxx.lanl.gov/abs/quant-ph/0403228

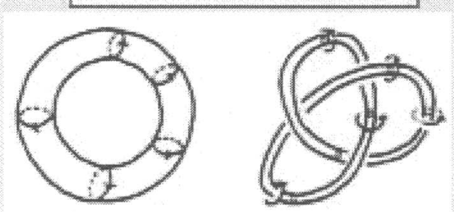
Lomonaco, Samuel J., Jr., The modern legacies of Thomson's atomic vortex theory in classical electrodynamics, AMS PSAPM/51, Providence, RI (1996), 145 - 166.

Kitaev, Alexei Yu, Fault-tolerant quantum computation by anyons, http://arxiv.org/abs/quant-ph/9707021

Rasetti, Mario, and Tullio Regge, Vortices in He II, current algebras and quantum knots, Physica 80 A, North-Holland, (1975), 217-2333.

What Motivated This Talk ?

Classical Vortices in Plasmas



Lomonaco, Samuel J., Jr., The modern legacies of Thomson's atomic vortex theory in classical electrodynamics, AMS PSAPM/51, Providence, RI (1996), 145 - 166.

Knots Naturally Arise in the Quantum World as Dynamical Processes

Examples of dynamical knots in quantum physics:
Knotted vortices

- In supercooled helium II
- In the Bose-Einstein Condensate
- In the Electron fluid found within the fractional quantum Hall effect

Reason for current intense interest:
A Natural Topological Obstruction to Decoherence

Objectives

- We seek to create a quantum system that simulates a closed knotted physical piece of rope.
- We seek to define a quantum knot in such a way as to represent the state of the knotted rope, i.e., the particular spatial configuration of the knot tied in the rope.
- We also seek to model the ways of moving the rope around (without cutting the rope, and without letting it pass through itself.)

Rules of the Game

Find a mathematical definition of a quantum knot that is

- Physically meaningful, i.e., physically implementable, and
- Simple enough to be workable and useable.

Aspirations

We would hope that this definition will be useful in modeling and predicting the behavior of knotted vortices that actually occur in quantum physics such as

- In supercooled helium II
- In the Bose-Einstein Condensate
- In the Electron fluid found within the fractional quantum Hall effect

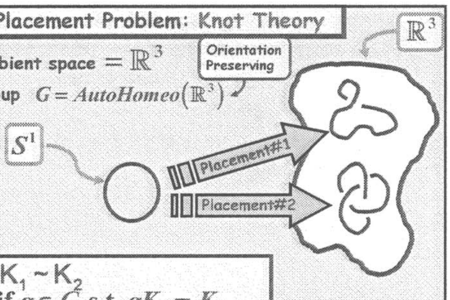
Quick Overview to Knot Theory

Mosaic knots

Placement Problem: Knot Theory

- Ambient space $= \mathbb{R}^3$
- Group $G = AutoHomeo(\mathbb{R}^3)$

Orientation Preserving



Def. $K_1 \sim K_2$ if $g \in G$ s.t. $gK_1 = K_2$

Problem. When are two placements the same ?

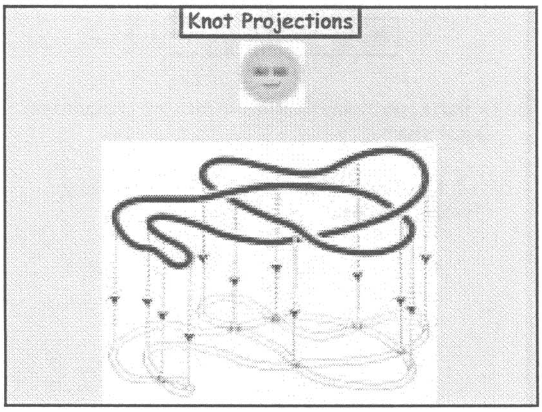
$K_1 \sim K_2$?

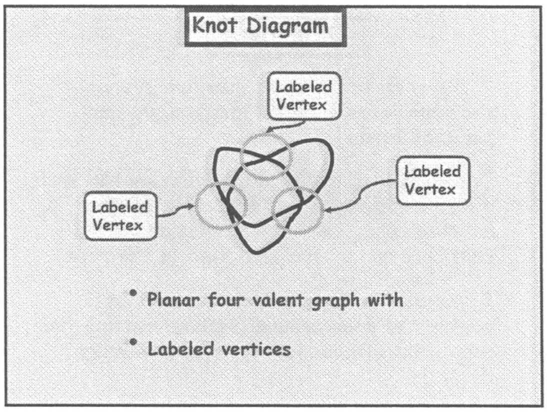
Equivalent Definition

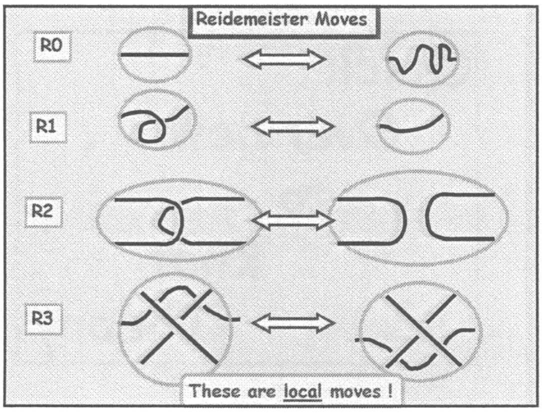
Def. $K_1 \sim K_2$ provided there exists a continuous family of auto-homeomorphisms

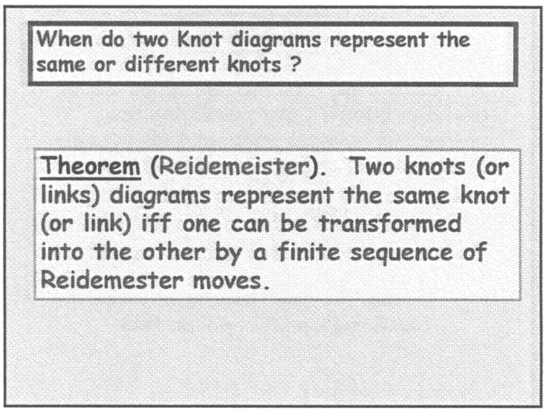
$$h_t : \mathbb{R}^3 \to \mathbb{R}^3 \quad (0 \le t \le 1)$$

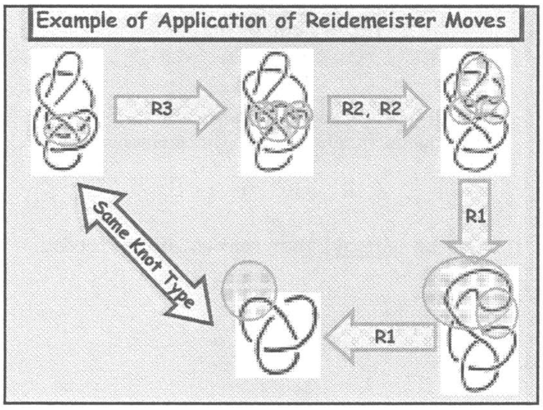
i.e., an isotopy, that continuously deforms K_1 into K_2.

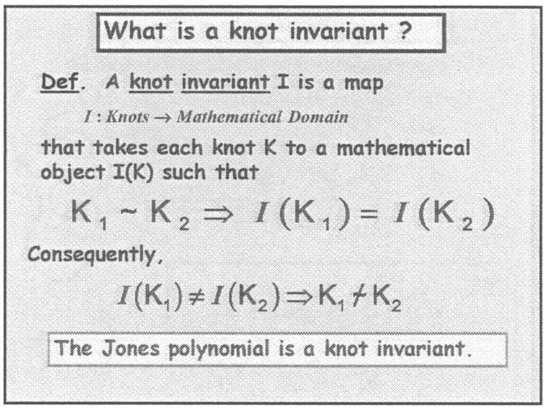

Knot Projections


Knot Diagram
Labeled Vertex
Labeled Vertex
Labeled Vertex
• Planar four valent graph with
• Labeled vertices


Reidemeister Moves
R0
R1
R2
R3
These are local moves !


When do two Knot diagrams represent the same or different knots ?
Theorem (Reidemeister). Two knots (or links) diagrams represent the same knot (or link) iff one can be transformed into the other by a finite sequence of Reidemester moves.


Example of Application of Reidemeister Moves
R3
R2, R2
R1
R1
Same Knot Type

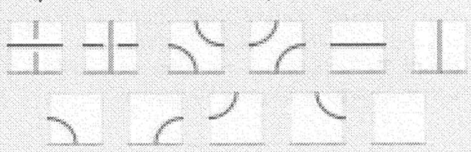

What is a knot invariant ?
Def. A knot invariant I is a map
$I : Knots \to Mathematical\ Domain$
that takes each knot K to a mathematical object I(K) such that
$K_1 \sim K_2 \Rightarrow I(K_1) = I(K_2)$
Consequently,
$I(K_1) \neq I(K_2) \Rightarrow K_1 \nsim K_2$
The Jones polynomial is a knot invariant.

Mosaic Knots

Lomonaco and Kauffman, Quantum Knots and Mosaics, Journal of Quantum Information Processing, vol. 7, Nos. 2-3, (2008), 85-115. An earlier version can be found at: http://arxiv.org/abs/0805.0339

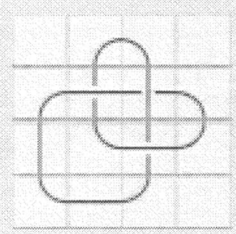
Mosaic Tiles

Let $T^{(u)}$ denote the following set of 11 symbols, called mosaic (unoriented) tiles:



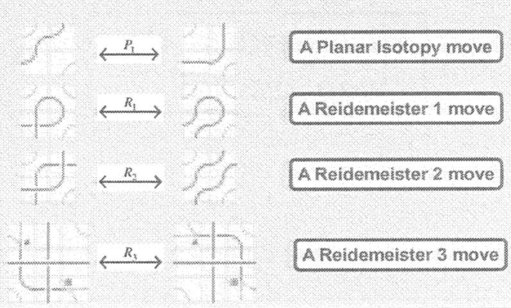
Please note that, up to rotation, there are exactly 5 tiles

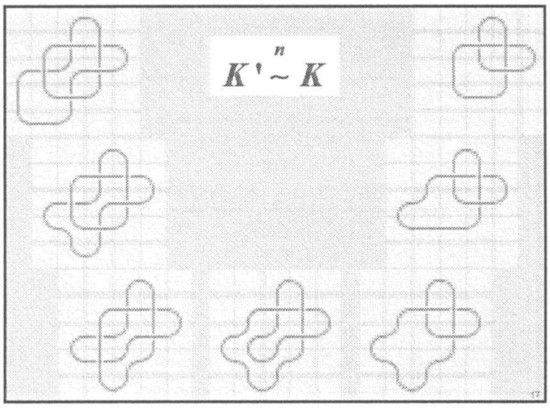
Mosaic Knots



A 4-mosaic trefoil

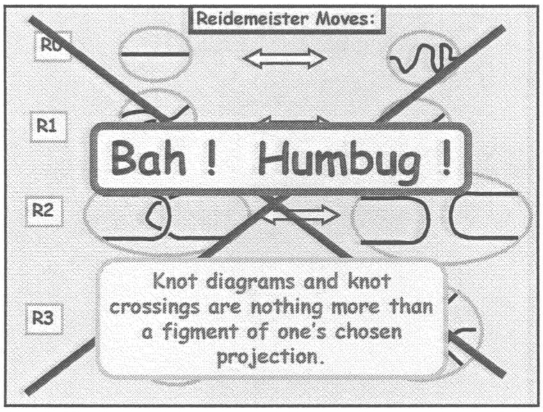
The Reidemeister and planar isotopy moves were reduced to a finite set of mosaic moves. Here are some examples:





But ...

But now we have a different but equivalent approach



Can we find an alternative approach to knot theory ?

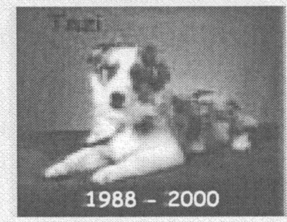
Knot projections have been around along time. They were used in the 1800s by Tait, Maxwell, Lord Kelvin. Can we move beyond knot diagrams and Reidemeister moves ? Today much more modern tools are available, such as 3-D graphics.

Can we find an alternate approach to knots that is much more "physics friendly" ???

Quantum Knots & Lattices
or,
How Wiggle, Wag, & Tug Go Quantum

How does a dog wag its tail ?

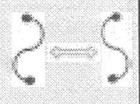
How does a dog wag its tail?



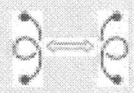
My best friend Tazi knew the answer.

How does a dog wag its tail ?

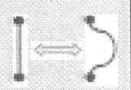
- She would wiggle her tail, just as a a creature would squirm on a flat planar surface.



- She would wag her tail in a twisting corkscrew motion.



- Her tail would also stretch or contract when an impolite child would tug on it.



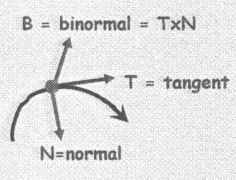
How does a dog wag its tail ?

Yes, when Tazi moved her tail, she naturally understood how a curve can move in 3-space !

She had a keen understanding of differential geometry.

Differential Geometry: The Frenet Frame

Each point of a curve in 3-space is naturally associated with a 3-frame, called the Frenet frame.



Differential Geometry: The Frenet Frame

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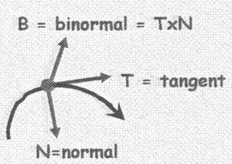
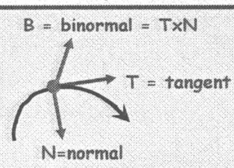


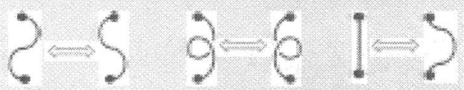
Differential Geometry: The Frenet Frame



- A curve bends by rotating about B – as measured by its curvature κ
- A curve twists by rotating about N – as measured by its torsion τ
- A curve stretches or contracts along its tangent T

Key Intuitive Idea

A curve in 3-space has 3 local (i.e., infinitesimal) degrees of freedom.



Wiggle
A curvature Move

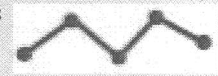
Wag
A torsion Move

Tug
A metric move

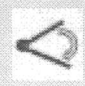
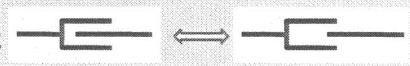
Can we take this idea and use it to create a useable well-defined set of moves which will replace the Reidemeister moves ?

Clues from Mechanical Engineering

Linkage = Inextensible bars (i.e., rods) connected by joints



Joints:

- Planar 
- Spherical 
- Slider 

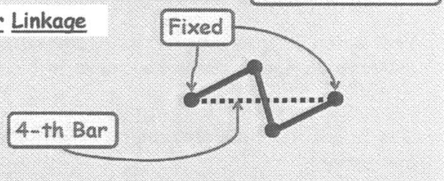
Clues from Mechanical Engineering

Mechanism = a linkage with 1 degree of freedom

Mechanisms

All Joints Planar

4-Bar Linkage



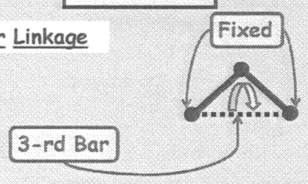
Since endpoints fixed, this is a local move on linkages. The rest of the linkage is untouched

This is a local curvature move, taking place in a fixed plane. We call it a wiggle.

Mechanisms

All Joints Spherical

3-Bar Linkage



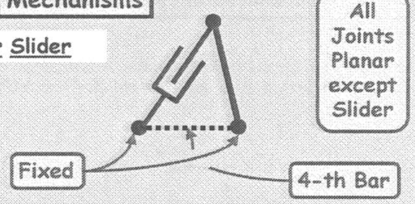
Since endpoints fixed, this is a local move on linkages. The rest of the linkage is untouched

This is a local torsion move, locally twisting the linkage into a new plane. We call it a wag.

Mechanisms

All Joints Planar except Slider

4-Bar Slider



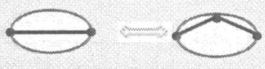
Since endpoints fixed, this is a local move on linkages. The rest of the linkage is untouched

This is a local expansion/contraction move, taking place in a fixed plane. We call it a tug.

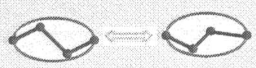
Translating M.E. into Knot Theory

Definition: Two piecewise linear (PL) knots K_1 and K_2 are said to be of the same knot type, written $K_1 \sim K_2$, provided one can be transformed into the other by a finite sequence of the following local moves:

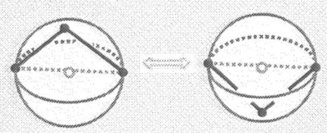
1) A tug:



2) A wiggle:



3) A wag:



Translating M.E. into Knot Theory

Using the methods found in Reidemeister's proof of the completeness of the Reidemeister moves, we have:

Theorem: Wiggles and wags can be expressed as sequences of tugs.

In fact, Reidemeister's fundamental move was essentially a tug.

So why bother with wiggles and wags ?

Why Wiggle & Wag?

My reason is that, while investigating electromagnetic knots, the knot theoretic tools I needed to study knots that naturally arise in physics were simply not available.

Lomonaco, Samuel J., Jr., The modern legacies of Thomson's atomic vortex theory in classical electrodynamics, AMS PSAPM/51, Providence, RI (1996), 145 - 166.

What was needed was a knot theory for inextensible knots. Reidemeister's moves, which are essentially derived from the tug move, are simply NOT inextensible moves.

Inextensible Knot Theory

Definition: Two piecewise linear (PL) knots K_1 and K_2 are said to be of the same inextensible knot type, written

$$K_1 \approx K_2 ,$$

provided that there exist subdivisions K'_1 and K'_2 of K_1 and K_2, respectively, such that K'_1 can be transformed into the K'_2 by a finite sequence of wiggles and wags.

Proposition. (Proof in progess) Let K_1 and K_2 be PL knots. Then

$$K_1 \approx K_2 \Leftrightarrow \begin{array}{c} K_1 \sim K_2 \\ \& \\ |K_1| = |K_2| \end{array}$$

Inextensible Knot Theory

Proposition. (Proof in progess) Let K_1 and K_2 be PL knots. Then

$$K_1 \approx K_2 \Leftrightarrow \begin{array}{c} K_1 \sim K_2 \\ \& \\ |K_1| = |K_2| \end{array}$$

So it would seem that we have gained NOTHING by creating inextensible knot theory !!!

But think again !

Inextensible Knot Theory

By working with this modified definition of knot type,

- We have lost none of the structure of classical knot theory.
- But we have succeeded in incorporating more of the geometry of 3-space.

Inextensible Knot Theory

Because of this modified definition, we will be able to:

- Create infinitesimal knot moves
- Create knot move differential forms
- Take variational derivatives with respect to these infinitesimal moves
- And much more

Lattice Knots

The Cubic Honeycomb
A Scaffolding for 3-Space

For each non-negative integer ℓ, let $\mathcal{L}_\ell$ denote the 3-D lattice of points

$$\mathcal{L}_\ell = \left\{ \left(\frac{m_1}{2^\ell}, \frac{m_2}{2^\ell}, \frac{m_3}{2^\ell} \right) : m_1, m_2, m_3 \in \mathbb{Z} \right\}$$

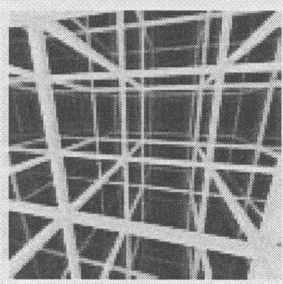
lying in Euclidean 3-space $\mathbb{R}^3$

This lattice determines a tiling of $\mathbb{R}^3$ by

$2^{-\ell} \times 2^{-\ell} \times 2^{-\ell}$ cubes,

called the <u>cubic honeycomb of</u> $\mathbb{R}^3$ <u>(of order</u> ℓ<u>)</u>

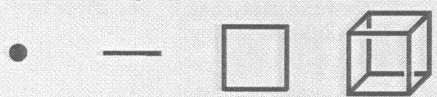
The Cubic Honeycomb (of order ℓ)
A Scaffolding for 3-Space



The Cubic Honeycomb

We think of this honeycomb as a cell complex $\mathcal{C}_\ell$ for $\mathbb{R}^3$ consisting of:

Vertices Edges Faces Cubes



$a \in \mathcal{L}_\ell$ E (Open) F (Open) B (Open)

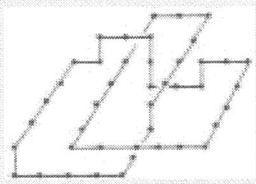
All cells of positive dimension are open cells.

Lattice Knots

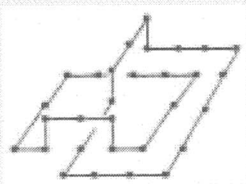
<u>Definition</u>. A <u>lattice graph</u> G <u>(of order</u> ℓ <u>)</u> is a finite subset of edges (together with their endpoints) of the honeycomb $\mathcal{C}_\ell$

<u>Definition</u>. A <u>lattice Knot</u> K <u>(of order</u> ℓ<u>)</u> is a lattice 2-valent graph (of order ℓ). Let $K^{(\ell)}$ denote the <u>set of all lattice knots of order</u> ℓ .

Lattice Knots



Lattice Trefoil

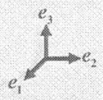


Lattice Hopf Link

Necessary Infrastructure

Orientation of 3-Space

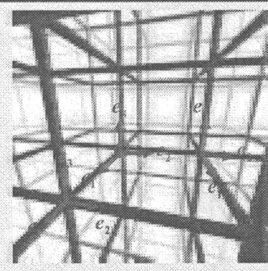
We define an orientation of $\mathbb{R}^3$ by selecting a right handed frame



at the origin O = (0,0,0) properly aligned with the edges of the honeycomb, and by parallel transporting it to each vertex $a \in L_\ell$

We refer to this frame as the <u>preferred frame</u>.

Orientation of 3-Space



The <u>preferred frame</u> at each lattice point.

Color Coding Conventions for Vertices & Edges

Solid Red	"Hollow" Gray	Solid Gray
● ▬	○ ┅	● ▬
Part of the Lattice Knot	Not part of Lattice Knot	Indeterminant, maybe part of Lattice Knot

A vertex a of a cube B is called a <u>preferred vertex</u> of B if the first octant of the preferred frame at a contains the cube B.

Since B is uniquely determined by its its preferred vertex, we use the notation

$$B = B^{(\ell)}(a)$$

The preferred edges and preferred faces of $B^{(\ell)}(a)$ are respectively the edges and faces of $B^{(\ell)}(a)$ that have a as a vertex

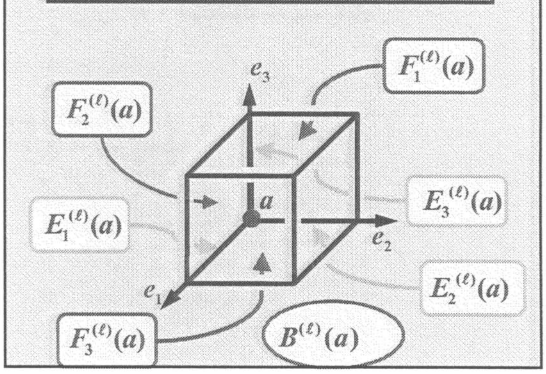
- Every edge is a preferred edge of exactly one cube
- Every face is the preferred face of exactly one cube

Hence, the following notation uniquely identifies each edge and face of the cell complex $\mathcal{C}_\ell$

$E_p^{(\ell)}(a) =$ Preferred edge parallel to e_p

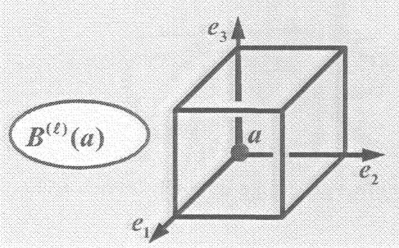
$F_p^{(\ell)}(a) =$ Preferred face perpendicular to e_p

Preferred Vertices, Edges, & Faces

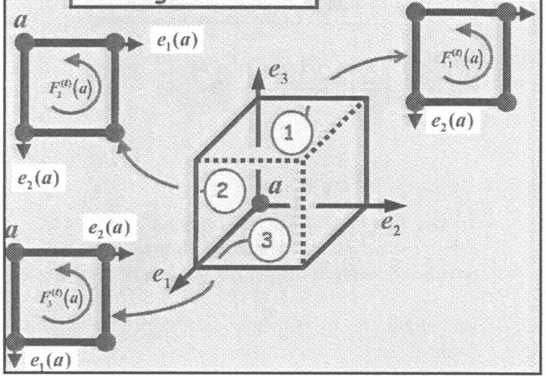


Drawing Conventions

When drawn in isolation, each cube $B^{(\ell)}(a)$ is drawn with edges parallel to the preferred frame, and with the preferred vertex in the back bottom left hand corner.



Drawing Conventions



The Left and Right Permutations

Define the left and right permutations ⌊ and ⌉ as

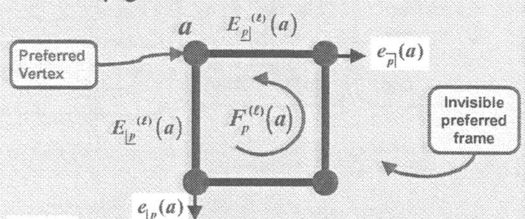
⌊ : {1,2,3}	→	{1,2,3}
1	↦	2
2	↦	3
3	↦	1

⌉ : {1,2,3}	→	{1,2,3}
1	↦	3
2	↦	1
3	↦	2

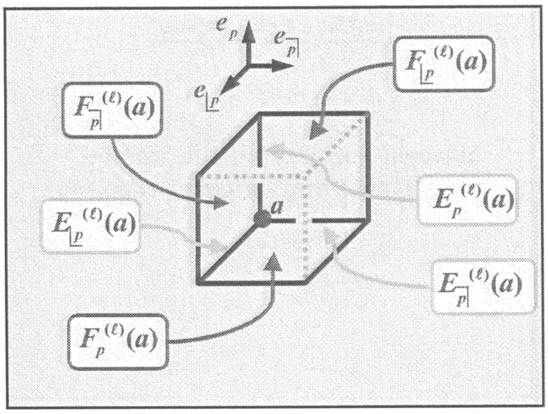
Ergo, $e_p = e_{\lfloor p} \times e_{p\rceil}$ $\quad e_{\lfloor p} = e_{p\rceil} \times e_p$ $\quad e_{p\rceil} = e_p \times e_{\lfloor p}$

Drawing Conventions

When drawn in isolation, $F_p^{(\ell)}(a)$ is always drawn with preferred vertex a in the upper left hand corner, and with $e_p(a)$ pointing out of the page.



$e_p(a)$ points out of the page toward the reader



Vertex Translation

Let a be a vertex in the lattice L_ℓ

$$a^{:p} = a + 2^{-\ell} e_p$$

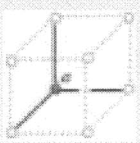
$$a^{:\bar{p}} = a - 2^{-\ell} e_p$$

$$a^{:p^3} = a + 3 \cdot 2^{-\ell} e_p$$

So for example,

$$a^{:1^2 \bar{2}^5 3} = a + 2 \cdot 2^{-\ell} e_1 - 5 \cdot 2^{-\ell} e_2 + 2^{-\ell} e_3$$

The Preferred Vertex (PV) Map $\lfloor - \rfloor_\ell$



$\lfloor - \rfloor_\ell$ → a

Preferred Vertex

$B^{(\ell)}(a) \cup \{a\} \cup \{E^{(\ell)}(a)\} \cup \{F^{(\ell)}(a)\}$

Half Closed Cube

$$\lfloor - \rfloor_\ell : \mathbb{R}^3 \to L_\ell$$

$$x = (x_1, x_2, x_3) \mapsto \left(2^{-\ell}\lfloor 2^\ell x_1 \rfloor, 2^{-\ell}\lfloor 2^\ell x_2 \rfloor, 2^{-\ell}\lfloor 2^\ell x_3 \rfloor\right)$$

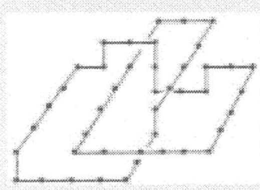
Lattice Knot Moves

Lattice Knots

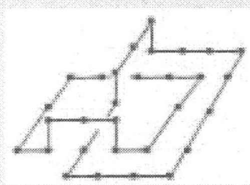
Definition. A lattice graph G (of order ℓ) is a finite subset of edges (together with their endpoints) of the honeycomb $\mathcal{C}_\ell$

Definition. A lattice Knot K (of order ℓ) is a lattice 2-valent graph (of order ℓ). Let $K^{(\ell)}$ denote the set of all lattice knots of order ℓ.

Lattice Knots



Lattice Trefoil



Lattice Hopf Link

Lattice knot moves

Definition. A lattice knot move μ (of order ℓ) is a bijection

$$\mu : \mathsf{K}^{(\ell)} \to \mathsf{K}^{(\ell)}$$

The move μ is said to be local if there exists a cube $B^{(\ell)}(a)$ in the lattice such that

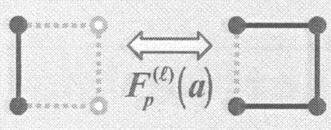
$$\mu\big|_{K-B^{(\ell)}(a)} = id\big|_{K-B^{(\ell)}(a)}$$

for all $K \in \mathsf{K}^{(\ell)}$

Lattice Knot Moves

We will now define the lattice knot moves tug, wiggle, and wag.

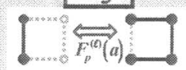
Tugs



$L_1^{(\ell)}(a, p, |⊐) = |⊐^{(\ell)}(a, p)$

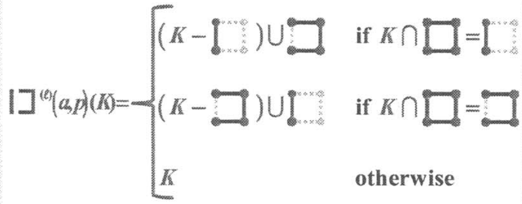
This is a local move on face $F_p^{(\ell)}(a)$

Tugs

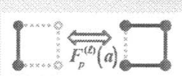


$L_1^{(\ell)}(a, p, |⊐) = |⊐^{(\ell)}(a, p)$

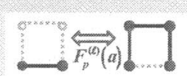
means



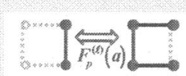
For each cube, 4 Tugs for each preferred face



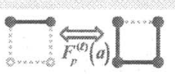
$|⊐^{(\ell)}(a,p)$



$⊔^{(\ell)}(a,p)$



$⊏|^{(\ell)}(a,p)$

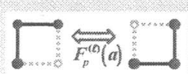


$⊓^{(\ell)}(a,p)$

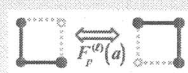
12 tugs for each cube

Tugs are extensible local moves

For each cube, 2 Wiggles for each preferred face



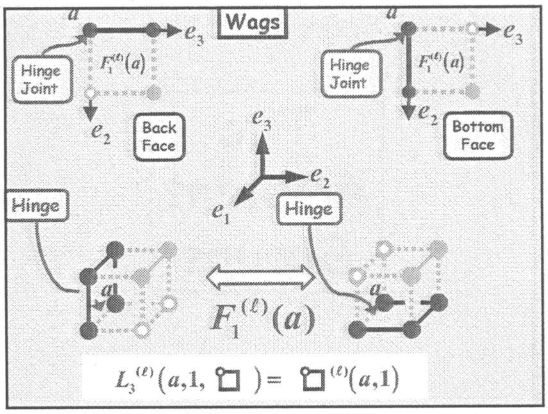
$L_2(a, p, ⊐) = ⊐^{(\ell)}(a,p)$



$L_2(a, p, ⊏) = ⊏^{(\ell)}(a,p)$

6 wiggles per cube

Wiggles are inextensible local moves



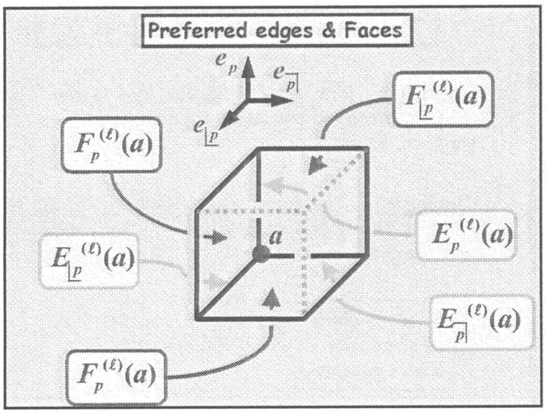
The Left and Right Permutations

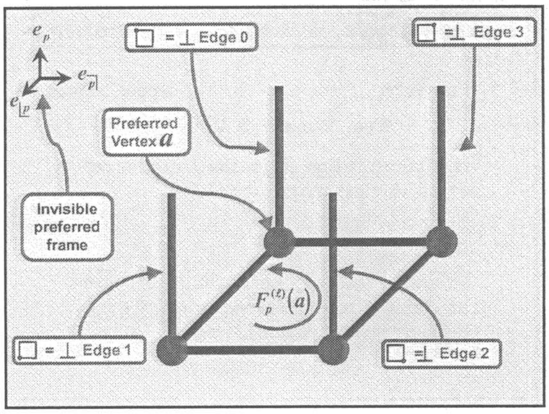
Please recall that the left and right permutations $\lfloor$ and $\rceil$ are defined as

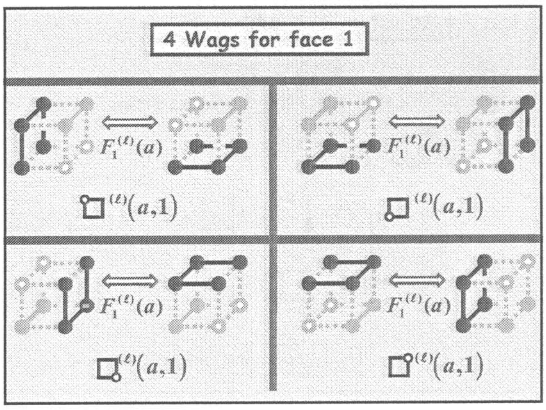
$\lfloor : \{1,2,3\}$	$\to$	$\{1,2,3\}$
1	$\mapsto$	2
2	$\mapsto$	3
3	$\mapsto$	1

$\rceil : \{1,2,3\}$	$\to$	$\{1,2,3\}$
1	$\mapsto$	3
2	$\mapsto$	1
3	$\mapsto$	2

$e_p = e_{\lfloor p} \times e_{p\rceil}$ $e_{\lfloor p} = e_{p\rceil} \times e_p$ $e_{p\rceil} = e_p \times e_{\lfloor p}$







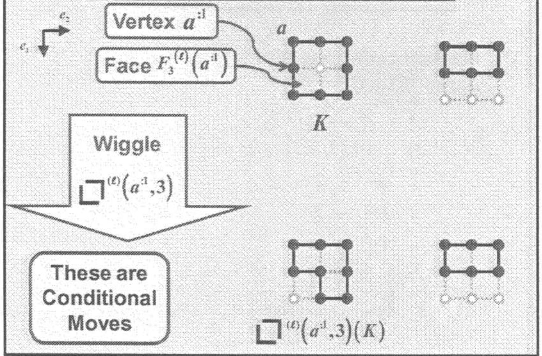
Wags

For each cube, there are 4 wags for each of its 3 preferred faces.

Hence, there are 12 wags per cube.

Wags are **inextensible** local moves

An Example of a Wiggle Move



The Ambient Groups

Tug, Wiggle, & Wag are Permutations

For each $\ell \geq 0$, each of the above moves,

Tug, Wiggle, & Wag,

is a permutation (bijection) on the set $K^{(\ell)}$ of all lattice knots of order ℓ .

In fact, each of the above local moves, as a permutation, is the product of disjoint transpositions.

The Ambient Groups Λ_ℓ and $\widetilde{\Lambda}_\ell$

Definition. The ambient group Λ_ℓ is the group generated by tugs, wiggles, and wags of order ℓ .

Definition. The inextensible ambient group $\widetilde{\Lambda}_\ell$ is the group generated only by wiggles and wags of order ℓ .

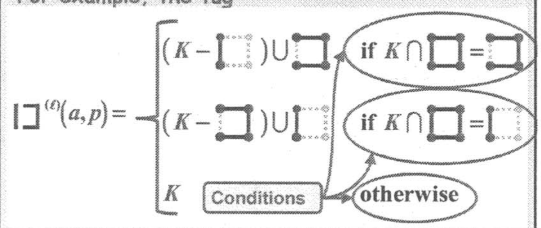
Tugs, wiggles, and wags are a set of involutions that generate the above groups.

What Is the Ambient Group ?

What is the ambient group ???

Observation: Wiggle, wag, and tug are symbolic conditional moves, as are the Reidemeister moves.

For example, the tug



What is the ambient group ???

Observation: Each is a symbolic representation of an authentic conditional move, i.e., a conditional orientation preserving (OP) auto-homeomorphism of $\mathbb{R}^3$.

Moreover, each involved (OP) auto-homeomorphism $h:\mathbb{R}^3 \to \mathbb{R}^3$ is local, i.e., there exists a 3-ball D such that

$$h|_{\mathbb{R}^3 - D} = id : \mathbb{R}^3 - D \to \mathbb{R}^3 - D$$

What is the ambient group ???

Let $LAH_{OP}(\mathbb{R}^3)$ the group of local OP auto-homeomorphisms of $\mathbb{R}^3$.

Let F be a family of knots in $\mathbb{R}^3$.

For example:

$K^{(\ell)}$ The family of lattice knots

S The family of finitely piecewise smooth (FPWS) knots in $\mathbb{R}^3$.

What is the ambient group ???

Def. A local authentic conditional (LAC) move on a family of knots F is a map

$$\begin{array}{rcl} \Phi: F & \to & LAH_{OP}(\mathbb{R}^3) \\ K & \mapsto & (\Phi_K : \mathbb{R}^3 \to \mathbb{R}^3) \end{array}$$

such that

$$\Phi_K(K) \in F \qquad \forall K \in F$$

Let $LAH_{OP}(\mathbb{R}^3)^F$ be the space of all LAC moves for the family F.

What is the ambient group ???

Let $LAH_{OP}(\mathbb{R}^3)^F$ be the space of all LAC moves for the family F.

Define a multiplication '•' as follows:

$$\begin{array}{rcl} LAH_{OP}(\mathbb{R}^3)^F \times LAH_{OP}(\mathbb{R}^3)^F & \to & LAH_{OP}(\mathbb{R}^3)^F \\ (\Phi', \Phi) & \mapsto & \Phi' \bullet \Phi \end{array}$$

as $(\Phi' \bullet \Phi)_K = \Phi'_{\Phi_K(K)} \circ \Phi_K$

where '∘' denotes the composition of functions.

What is the ambient group ???

Proposition. $\left(LAH_{OP}(\mathbb{R}^3)^F, \bullet\right)$ is a monoid.

In the paper "Quantum Knots and Lattices," we construct a faithful representation

$$\Gamma : \Lambda_\ell \to LAH_{OP}(\mathbb{R}^3)^F$$

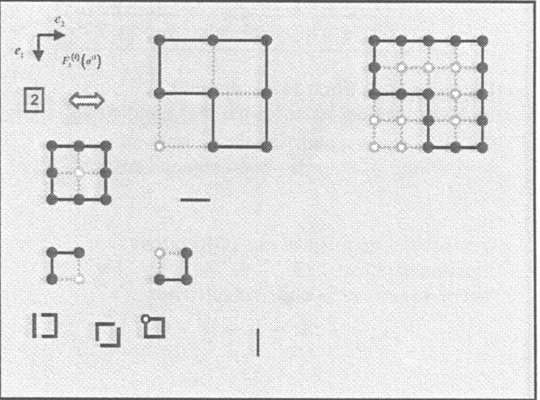
into a subgroup of the moniod $\left(LAH_{OP}(\mathbb{R}^3)^F, \bullet\right)$ by mapping each generator wiggle, wag, and tug onto a local conditional OP auto-homeomorphism of $\mathbb{R}^3$.

Refinement

The refinement injection

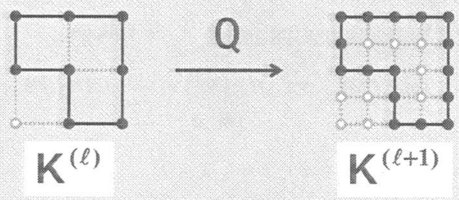
Def. We define the refinement injection $Q: K^{(\ell)} \to K^{(\ell+1)}$ from lattice knots of order ℓ to lattice knots of order $\ell+1$ as

$$\begin{array}{llll} Q: & K^{(\ell)} & \to & K^{(\ell+1)} \\ & K & \mapsto & \bigcup_{a \in L_\ell} \bigcup_{p=1}^{3} \bigcup_{E_p^{(\ell)}(a)} \left\{ \overline{E_p^{(\ell)}(a)}, \overline{E_p^{(\ell)}(a^{:p})} \right\} \end{array}$$



The refinement injection

An example:



Conjectured Refinement Monomorphism

We conjecture the existence of a refinement monomorphism

$$Q: \Lambda_\ell \to \Lambda_{\ell+1}$$

which preserves the action

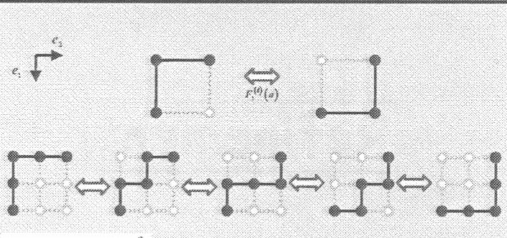
$$\begin{array}{ccc} \Lambda_\ell \times K^{(\ell)} & \to & K^{(\ell)} \\ (g, K) & \mapsto & gK \end{array}$$

i.e., with the property

$$Q(g)Q(K) = Q(gK)$$

In fact, we have a construction which we believe is such a monomorphism.

The Refinement Morphism $Q: \Lambda_\ell \to \Lambda_{\ell+1}$???



$$Q\left(\square^{(\ell)}(a,1)\right) \stackrel{?}{=}$$

$$\square^{(\ell+1)}\left(a^{:23},1\right)\square^{(\ell+1)}\left(a^{:2},1\right)\square^{(\ell+1)}\left(a^{:3},1\right)\square^{(\ell+1)}\left(a,1\right)$$

Knot Type

Lattice Knot Type

Two lattice knots K_1 and K_2 in $\mathrm{K}^{(\ell)}$ are of the same ℓ-type, written

$$K_1 \underset{\ell}{\sim} K_2$$

provided there is an element $g \in \Lambda_\ell$ such that

$$gK_1 = K_2$$

They are of the same knot type, written

$$K_1 \sim K_2$$

provided there is a non-negative integer m such that

$$\mathrm{Q}^m K_1 \underset{\ell+m}{\sim} \mathrm{Q}^m K_2$$

Inextensible Lattice Knot Type

Two lattice knots K_1 and K_2 in $\mathrm{K}^{(\ell)}$ are of the same inextensible ℓ-type, written

$$K_1 \underset{\ell}{\approx} K_2$$

provided there is an element $g \in \widetilde{\Lambda}_\ell$ such that

$$gK_1 = K_2$$

They are of the same inextensible knot type, written

$$K_1 \approx K_2$$

provided there is a non-negative integer m such that

$$\mathrm{Q}^m K_1 \underset{\ell+m}{\approx} \mathrm{Q}^m K_2$$

n-Bounded Lattices, Lattice knots, and Ambient Groups

In preparation for creating a definition of physically emplementable quantum knot systems, we need to work with finite mathematical objects.

n-Bounded Lattices

Let ℓ and n be be non-negative integers. We define the n-bounded lattice of order ℓ as

$$L_{\ell,n} = \{ a \in L_\ell : |a|_\infty \le n \}$$

where

$$|a|_\infty = \max_j (|a_j|)$$

We also have

$C_{\ell,n}$ The corresponding cell complex

$C^j_{\ell,n}$ The corresponding j-skeleton

n-Bounded Lattice Knots & Ambient Groups

$\mathrm{K}^{(\ell,n)} = \mathrm{K}^{(\ell)} \cap L_{\ell,n}$ set of n-bounded lattice knots of order ℓ

$\Lambda_{\ell,n} = \Lambda_\ell |_{C_{\ell,n}}$ Ambient group of order (ℓ,n)

$\widetilde{\Lambda}_{\ell,n} = \widetilde{\Lambda}_\ell |_{C_{\ell,n}}$ Inextensible Ambient group of order (ℓ,n)

n-Bounded Lattice Knots & Ambient Groups

We also have the injection

$$\iota : \mathsf{K}^{(\ell,n)} \to \mathsf{K}^{(\ell,n+1)}$$

and the monomorphisms

$$\iota : \Lambda^{(\ell,n)} \to \Lambda^{(\ell,n+1)} \quad \text{and} \quad \iota : \widetilde{\Lambda}^{(\ell,n)} \to \widetilde{\Lambda}^{(\ell,n+1)}$$

We thus have a nested sequence of lattice knot systems

$$\left(\mathsf{K}^{(\ell,1)},\Lambda_{\ell,1}\right)\to\left(\mathsf{K}^{(\ell,2)},\Lambda_{\ell,2}\right)\to\cdots\to\left(\mathsf{K}^{(\ell,n)},\Lambda_{\ell,n}\right)\to\cdots$$

n-Bounded Lattice Knot Type

Two lattice knots K_1 and K_2 in $\mathsf{K}^{(\ell,n)}$ are said to be of the <u>same</u> <u>lattice</u> <u>knot</u> (ℓ,n) -<u>type</u>, written

$$K_1 \underset{\ell}{\overset{n}{\sim}} K_2$$

provided there is an element $g \in \Lambda_{\ell,n}$ such that

$$gK_1 = K_2$$

They are of the <u>same</u> <u>lattice</u> <u>knot</u> <u>type</u>, written $K_1 \sim K_2$

provided there are non-negative integers ℓ' and n' such that

$$\iota^{n'} Q^{\ell'} K_1 \underset{\ell+\ell'}{\overset{n+n'}{\sim}} \iota^{n'} Q^{\ell'} K_2$$

n-Bounded Lattice Knot Type

In like manner for the inextensible ambient group $\widetilde{\Lambda}_{\ell,n}$, we can define

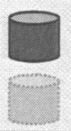
$$K_1 \underset{\ell}{\overset{n}{\approx}} K_2 \quad \text{and} \quad K_1 \approx K_2$$

Quantum Knots & Quantum Knot Systems

Quantum Knots

It's time to remodel the bounded lattice $\mathsf{L}_{\ell,n}$ by painting all its edges.

Two available cans of paint



"Solid" Red ———— An Edge

"Hollow" Gray ············ A Non-Edge

Set of all 2-colorings of edges of $\mathsf{L}_{\ell,n}$ ⟵ Identification ⟶ Set $\mathsf{K}^{(\ell,n)}$ of all lattice graphs in $\mathsf{L}_{\ell,n}$

Quantum Knots

E Edge Coloring Space

E = 2-D Hilbert space with orthonormal basis

$|0\rangle = |\ \cdots\cdots\cdots\ \rangle$ — <u>Non-Edge</u> — "Hollow" Gray

$|1\rangle = |\ \text{———}\ \rangle$ — <u>Existent Edge</u> — "Solid" Red

$\mathsf{G}^{(\ell,n)}$ Hilbert Space of Lattice Graphs in $\mathsf{L}_{\ell,n}$

$$\mathsf{G}^{(\ell,n)} = \bigotimes_{E^{(\ell)} \in Edges(\mathsf{L}_{\ell,n})} E$$

Quantum Knots

$\mathcal{G}^{(\ell,n)}$ Hilbert Space of Lattice Graphs in $\mathcal{L}_{\ell,n}$

$$\mathcal{G}^{(\ell,n)} = \bigotimes_{E^{(\ell)} \in Edges(\mathcal{L}_{\ell,n})} \mathcal{E}$$

Orthonormal basis is:

$$\left\{ \bigotimes_{E^{(\ell)} \in Edges(\mathcal{L}_{\ell,n})} \left| E^{(\ell)}, c\left(E^{(\ell)}\right) \right\rangle \;\middle|\; c : Edges(\mathcal{L}_{\ell,n}) \to \{\text{Gray}, \text{Red}\} \right\}$$

which is identified with

$$\{ \; |G\rangle \; | \; G \text{ lattice graph in } \mathcal{L}_{\ell,n} \; \}$$

Quantum Knots

$\mathcal{G}^{(\ell,n)}$ Hilbert Space of Lattice Graphs in $\mathcal{L}_{\ell,n}$

$$\mathcal{G}^{(\ell,n)} = \bigotimes_{E^{(\ell)} \in Edges(\mathcal{L}_{\ell,n})} \mathcal{E}$$

Orthonormal basis is:

$$\{ \; |G\rangle \; | \; G \text{ lattice graph in } \mathcal{L}_{\ell,n} \; \}$$

Quantum Knots

$\mathcal{G}^{(\ell,n)}$ Hilbert Space of Lattice Graphs in $\mathcal{L}_{\ell,n}$

$$\mathcal{G}^{(\ell,n)} = \bigotimes_{E^{(\ell)} \in Edges(\mathcal{L}_{\ell,n})} \mathcal{E}$$

Orthonormal basis is:

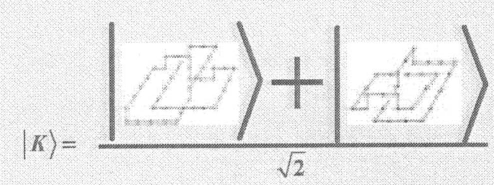
$$\{ \; |G\rangle \; | \; G \text{ lattice graph in } \mathcal{L}_{\ell,n} \; \}$$

$\mathcal{K}^{(\ell,n)}$ Hilbert Space of quantum knots

$\mathcal{K}^{(\ell,n)}$ = Sub-Hilbert space of $\mathcal{G}^{(\ell,n)}$ with orthonormal basis

$$\{ |K\rangle \mid K \in \mathcal{K}^{(\ell,k)} \}$$

An Example of a Quantum Knot



The Ambient Group $\Lambda_{\ell,n}$ as a Unitary Group

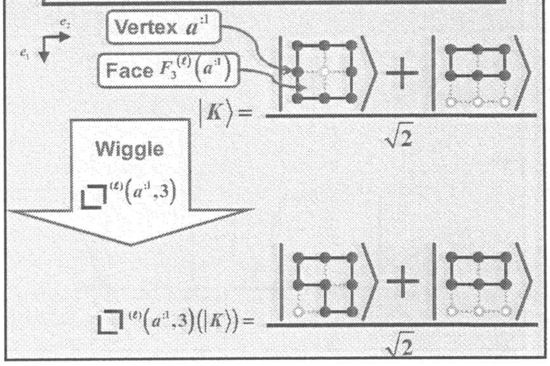
We identify each element $g \in \Lambda_{\ell,n}$ with the linear transformation defined by

$$\begin{array}{ccc} \mathcal{K}^{(\ell,n)} & \to & \mathcal{K}^{(\ell,n)} \\ |K\rangle & \mapsto & |gK\rangle \end{array}$$

Since each element $g \in \Lambda_{\ell,n}$ is a permutation, it is a linear transformation that simply permutes basis elements.

Hence, under this identification, the ambient group $\Lambda_{\ell,n}$ becomes a discrete group of unitary transfs on the Hilbert space $\mathcal{K}^{(\ell,n)}$.

An Example of the $\Lambda_{\ell,n}$ Group Action



The Quantum Knot System $\left(K^{(\ell,n)},\Lambda_{\ell,n}\right)$

Def. A quantum knot system $\left(K^{(\ell,n)},\Lambda_{\ell,n}\right)$ is a quantum system having $K^{(\ell,n)}$ as its state space, and having the Ambient group $\Lambda_{\ell,n}$ as its set of accessible unitary transformations.

The states of quantum system $\left(K^{(\ell,n)},\Lambda_{\ell,n}\right)$ are quantum knots. The elements of the ambient group $A(n)$ are quantum moves.

$$\left(K^{(\ell,1)},\Lambda_{\ell,1}\right)\xrightarrow{\iota}\cdots\xrightarrow{\iota}\left(K^{(\ell,n)},\Lambda_{\ell,n}\right)\xrightarrow{\iota}\left(K^{(\ell,n+1)},\Lambda_{\ell,n+1}\right)\xrightarrow{\iota}\cdots$$

Physically Implementable — Physically Implementable — Physically Implementable

The Quantum Knot System $\left(K^{(\ell,n)},\Lambda_{\ell,n}\right)$

$$\left(K^{(\ell,1)},\Lambda_{\ell,n}\right)\xrightarrow{\iota}\cdots\xrightarrow{\iota}\left(K^{(\ell,n)},\Lambda_{\ell,n}\right)\xrightarrow{\iota}\left(K^{(\ell,n+1)},\Lambda_{\ell,n+1}\right)\xrightarrow{\iota}\cdots$$

Physically Implementable — Physically Implementable — Physically Implementable

Choosing integers ℓ and n is analogous to choosing respectively the thickness and the length of the rope. The smaller the thickness and the longer the rope, the more knots that can be tied.

The parameters (wiggle, wag, & tug) of the ambient group $\Lambda_{\ell,n}$ are the "knobs" one turns to spacially manipulate the quantum knot.

Quantum Knot Type

Def. Two quantum knots $|K_1\rangle$ and $|K_2\rangle$ are of the same knot (ℓ,n)-type, written

$$|K_1\rangle \underset{\ell}{\overset{n}{\sim}} |K_2\rangle,$$

provided there is an element $g\in\Lambda_{\ell,n}$ s.t.

$$g|K_1\rangle=|K_2\rangle$$

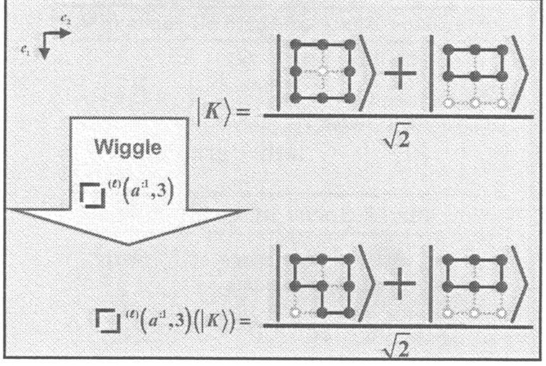
They are of the same knot type, written

$$|K_1\rangle\sim|K_2\rangle,$$

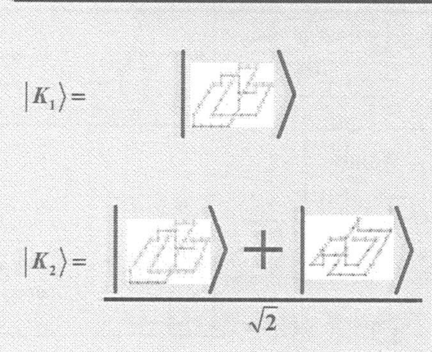
provided there are integer $\ell',n'\geq 0$ such that

$$Q^{\ell'}\iota^{n'}|K_1\rangle \underset{\ell+\ell'}{\overset{n+n'}{\sim}} Q^{\ell'}\iota^{n'}|K_2\rangle$$

Two Quantum Knots of the Same Knot Type



Two Quantum Knots NOT of the Same Knot Type



Hamiltonians
of the
Generators
of the
Ambient Group

Hamiltonians for $A(n)$

Each generator $g \in \Lambda_{\ell,n}$ is the product of disjoint transpositions, i.e.,

$$g = \left(K_{\alpha_1}, K_{\beta_1}\right)\left(K_{\alpha_2}, K_{\beta_2}\right)\cdots\left(K_{\alpha_r}, K_{\beta_r}\right)$$

Choose a permutation η so that

$$\eta^{-1} g \eta = \left(K_1, K_2\right)\left(K_3, K_4\right)\cdots\left(K_{r-1}, K_r\right)$$

Hence,

$$\eta^{-1} g \eta = \begin{pmatrix} \sigma_1 & & & & \\ & \sigma_1 & & \mathbf{0} & \\ & & \ddots & & \\ & \mathbf{0} & & \sigma_1 & \\ & & & & I_{d(\ell,n)-2r} \end{pmatrix}, \text{ where } \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$d(\ell,n) = \dim\left(\mathrm{K}^{(\ell,n)}\right)$

Hamiltonians for $A(n)$

Also, let $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and note that

$$\ln(\sigma_1) = \frac{i\pi}{2}(2s+1)(\sigma_0 - \sigma_1), \quad s \in \mathbb{Z}$$

For simplicity, we always choose the branch $s = 0$.

$$H_g = -i\eta \ln\left(\eta^{-1} g \eta\right)\eta^{-1}$$

$$= \frac{\pi}{2}\eta \begin{pmatrix} I_r \otimes (\sigma_0 - \sigma_1) & 0 \\ 0 & 0_{(d(\ell,n)-2r)\times(d(\ell,n)-2r)} \end{pmatrix}\eta^{-1}$$

Matrix log def

The Log of a Unitary Matrix

Let U be an arbitrary finite rxr unitary matrix.

Then eigenvalues of U all lie on the unit circle in the complex plane.

Moreover, there exists a unitary matrix W which diagonalizes U, i.e., there exists a unitary matrix W such that

$$WUW^{-1} = \Delta\left(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_r}\right)$$

where $e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_r}$ are the eigenvalues of U.

The Log of a Unitary Matrix

Then

$$\ln(U) = W^{-1}\Delta\left(\ln(e^{i\theta_1}), \ln(e^{i\theta_2}), \ldots, \ln(e^{i\theta_r})\right)W$$

Since $\ln\left(e^{i\theta_j}\right) = i\theta_j + 2\pi i n_j$, where $n_j \in \mathbb{Z}$ is an arbitrary integer, we have

$$\ln(U) = iW^{-1}\Delta\left(\theta_1 + 2\pi n_1, \theta_2 + 2\pi n_2, \ldots, \theta_r + 2\pi n_r\right)W$$

where $n_1, n_2, \ldots, n_r \in \mathbb{Z}$

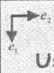
The Log of a Unitary Matrix

Since $e^A = \sum_{m=0}^{\infty} A^m / (m!)$, we have

$$\begin{aligned} e^{\ln(U)} &= e^{W^{-1}\Delta(\ln i\theta_1, \ldots, \ln i\theta_r)W} \\ &= W^{-1} e^{\Delta(\ln i\theta_1, \ldots, \ln i\theta_r)} W \\ &= W^{-1}\Delta\left(e^{\ln i\theta_1}, \ldots, e^{\ln i\theta_r}\right)W \\ &= W^{-1}\Delta\left(e^{i\theta_1 + 2\pi i n_1}, \ldots, e^{i\theta_r + 2\pi i n_r}\right)W \\ &= W^{-1}\Delta\left(e^{i\theta_1}, \ldots, e^{i\theta_r}\right)W = U \end{aligned}$$

Back

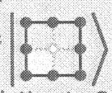
Hamiltonians for $\Lambda_{\ell,n}$



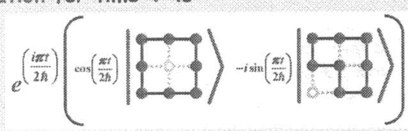
$F_3^{(\ell)}\left(a^{:1}\right)$

Using the Hamiltonian for the wiggle move

and the initial state



we have that the solution to Schroedinger's equation for time t is



Observables which are Quantum Knot Invariants

Observable Q. Knot Invariants

Question. What do we mean by a physically observable knot invariant ?

Let $\left(\mathrm{K}^{(\ell,n)},\Lambda_{\ell,n}\right)$ be a quantum knot system. Then a quantum observable Ω is a Hermitian operator on the Hilbert space $\mathrm{K}^{(\ell,n)}$.

Observable Q. Knot Invariants

Question. But which observables Ω are actually knot invariants ?

Def. An observable Ω is an invariant of quantum knots provided $U\Omega U^{-1}=\Omega$ for all $U\in\Lambda_{\ell,n}$

Observable Q. Knot Invariants

Question. But how do we find quantum knot invariant observables ?

Theorem. Let $\left(\mathrm{K}^{(\ell,n)},\Lambda_{\ell,n}\right)$ be a quantum knot system, and let

$$\mathrm{K}^{(\ell,n)}=\bigoplus_r W_r$$

be a decomposition of the representation

$$\Lambda_{\ell,n}\times\mathrm{K}^{(\ell,n)}\to\mathrm{K}^{(\ell,n)}$$

into irreducible representations .

Then, for each r, the projection operator P_r for the subspace W_r is a quantum knot observable.

Observable Q. Knot Invariants

Theorem. Let $\left(\mathrm{K}^{(\ell,n)},\Lambda_{\ell,n}\right)$ be a quantum knot system, and let Ω be an observable on $\mathrm{K}^{(\ell,n)}$. Let $St(\Omega)$ be the stabilizer subgroup for Ω , i.e.,

$$St(\Omega)=\left\{U\in A(n)\ :\ U\Omega U^{-1}=\Omega\right\}$$

Then the observable

$$\sum_{U\in\Lambda_{\ell,n}/St(\Omega)} U\Omega U^{-1}$$

is a quantum knot invariant, where the above sum is over a complete set of coset representatives of $St(\Omega)$ in $\Lambda_{\ell,n}$.

Observable Q. Knot Invariants

In $\mathrm{K}^{(\ell,n)}$, the following is an example of an inextensible quantum knot invariant observable:

$$\Omega=\sum_{p=1}^{3}\left|\partial F_p^{(\ell)}(a)\right\rangle\left\langle\partial F_p^{(\ell)}(a)\right|+\sum_{p=1}^{3}\left|\partial F_p^{(\ell)}(a^{:p})\right\rangle\left\langle\partial F_p^{(\ell)}(a^{:p})\right|$$

where $\partial F_p^{(\ell)}(a)$ denotes the boundary of the face $F_p^{(\ell)}(a)$.

Future Directions & Open Questions

Future Directions & Open Questions

- What is the structure of the ambient groups Λ_ℓ, $\widetilde{\Lambda}_\ell$, $\Lambda_{\ell,n}$, $\widetilde{\Lambda}_{\ell,n}$, and their direct limits ? Can one find a presentation of these groups ? Are they Coxeter groups?

- Exactly how are the lattice and the mosaic ambient groups related to one another.

Future Directions & Open Questions

- Unlike classical knots, quantum knots can exhibit the non-classical behavior of quantum superposition and quantum entanglement. Are quantum and topological entanglement related to one another ? If so, how ?

Future Directions & Open Questions

- How does one find a quantum observable for the Jones polynomial ? This would be a family of observables parameterized by points on the circle in the complex plane. Does this approach lead to an algorithmic improvement to the quantum algorithm created by Aharonov, Jones, and Landau ?

- How does one create quantum knot observables that represent other knot invariants such as, for example, the Vassiliev invariants ?

Future Directions & Open Questions

- What is gained by extending the definition of quantum knot observables to POVMs ?

- What is gained by extending the definition of quantum knot observables to mixed ensembles ?

Future Directions & Open Questions

Def. We define the lattice number of a knot K as the smallest integer n for which K is representable as a lattice knot of order $(\ell = 0, n)$

How does one compute the lattice number of a knot? How does one find a quantum observable for the lattice number?

How is the lattice number related to the mosaic number of a knot?

Future Directions & Open Questions

Quantum Knot Tomography: Given many copies of the same quantum knot, find the most efficient set of measurements that will determine the quantum knot to a chosen tolerance $\varepsilon > 0$.

Quantum Braids: Use lattices to define quantum braids. How are such quantum braids related to the work of Freedman, Kitaev, et al on anyons and topological quantum computing?

Future Directions & Open Questions

- Can quantum knot systems be used to model and predict the behavior of
  - Quantum vortices in supercooled helium 2 ?
  - Quantum vortices in the Bose-Einstein Condensate
  - Fractional charge quantification that is manifest in the fractional quantum Hall effect

UMBC Quantum Knots Research Lab

We at UMBC are very proud of our new state of the art Quantum Knots Research Laboratory.

We have just purchased some of the latest and most advanced equipment in quantum knots research !!!