Knot Module Lecture Notes (as of June 2005)
Day 1 - Crossing and Linking number

1. Definition and crossing number. (Reference: §§1.1 & 3.3 of [A]. References are listed at the end of these notes.)

   - Introduce idea of knot using physical representations (e.g., extension cord, rope, wire, chain, tangle toys).
   - Need to connect ends or else can undo knot; we're allowed to do anything but cut.
   - The simplest knot is the unknot, but even this can look quite complicated. (Before class, scrunch up a big elastic band to make it look complicated. Then, in class, unravel it to illustrate that it's the same as the unknot).
   - The next is the trefoil. There are two fairly simple projections which look quite different (demonstrate with extension cord).

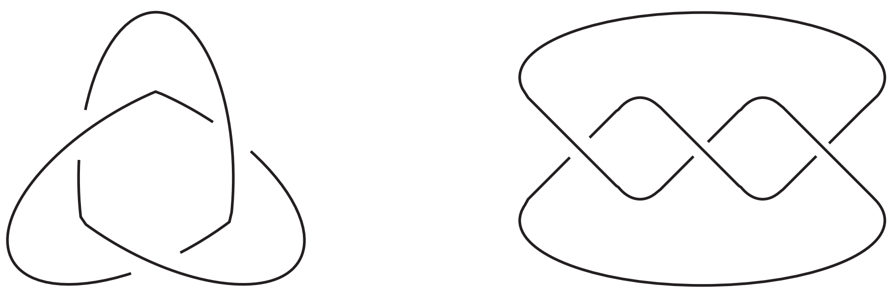


Figure 1: Two projections of the trefoil.

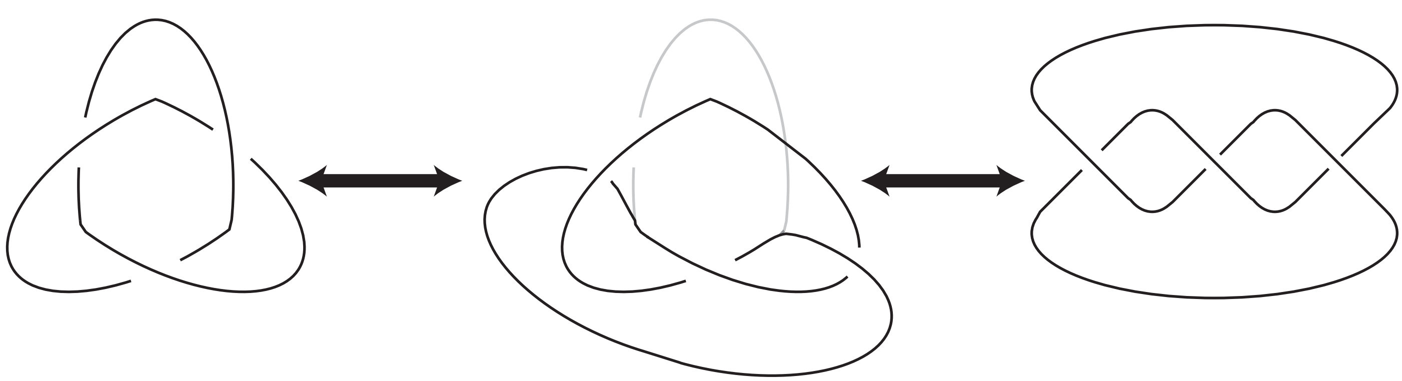


Figure 2: How to get from one projection to the other.

   - We say the trefoil has crossing number three (we will write $C(\text{trefoil}) = 3$), since it has no projection with fewer than three crossings. We will prove this by showing projections of 0, 1, or 2 crossings are the unknot.

     Show that projections of 1 crossing are all the unknot. You may wish to identify projections that are the same up to rotation. In this case, there are four types of

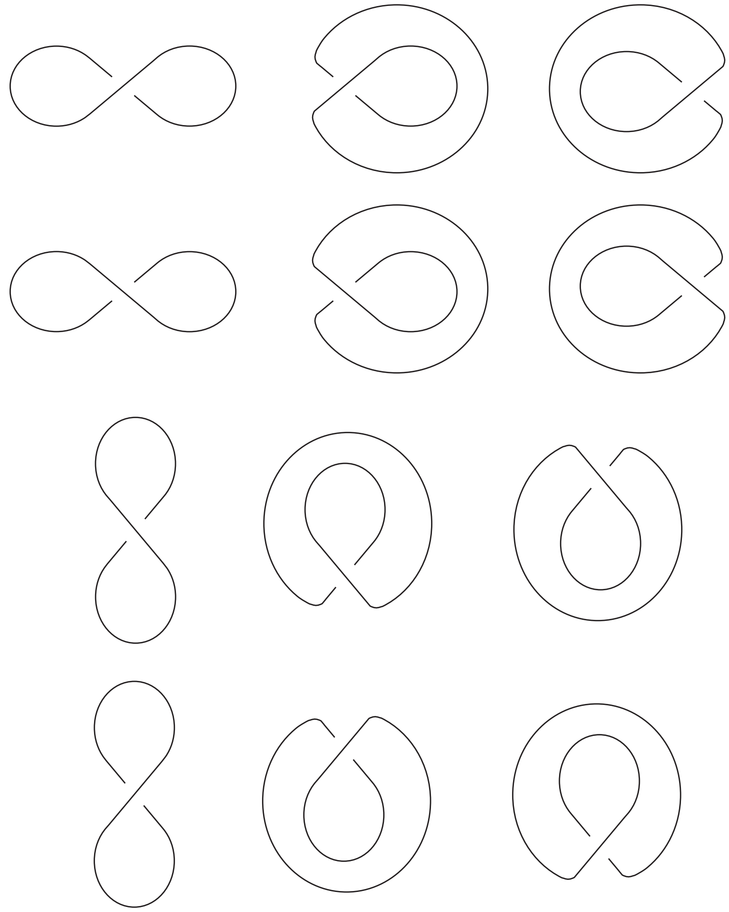


Figure 3: Some 1-crossing projections.

1-crossing projection. If you also allow reflections, then there are only two types, one represented by the left column of Figure 3 and the other by any of the eight other projections.

The crossing number two case is homework.

- There are also two different flavours of trefoil, left and right. They are mirror reflections of one another.

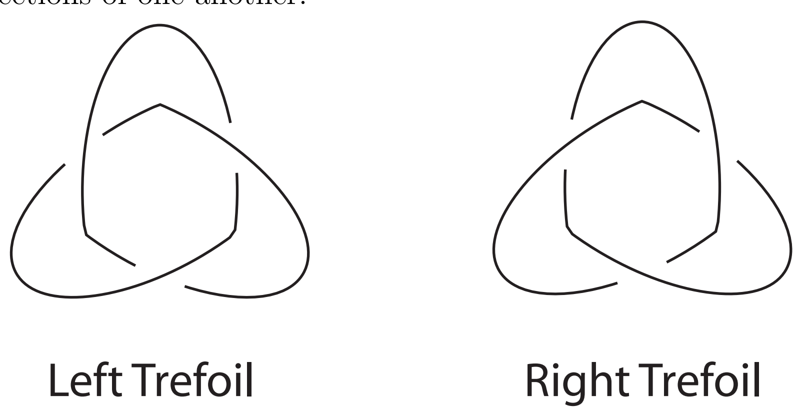


Figure 4: Left and Right trefoils.

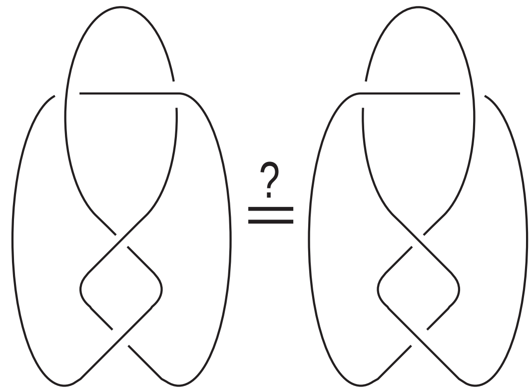


Figure 5: Is the figure 8 the same as its mirror reflection?

This doesn't happen for next most complicated knot, the figure eight, since it's its own mirror reflection.

Allow students to explore going between different projections of unknot, trefoil and figure eight.

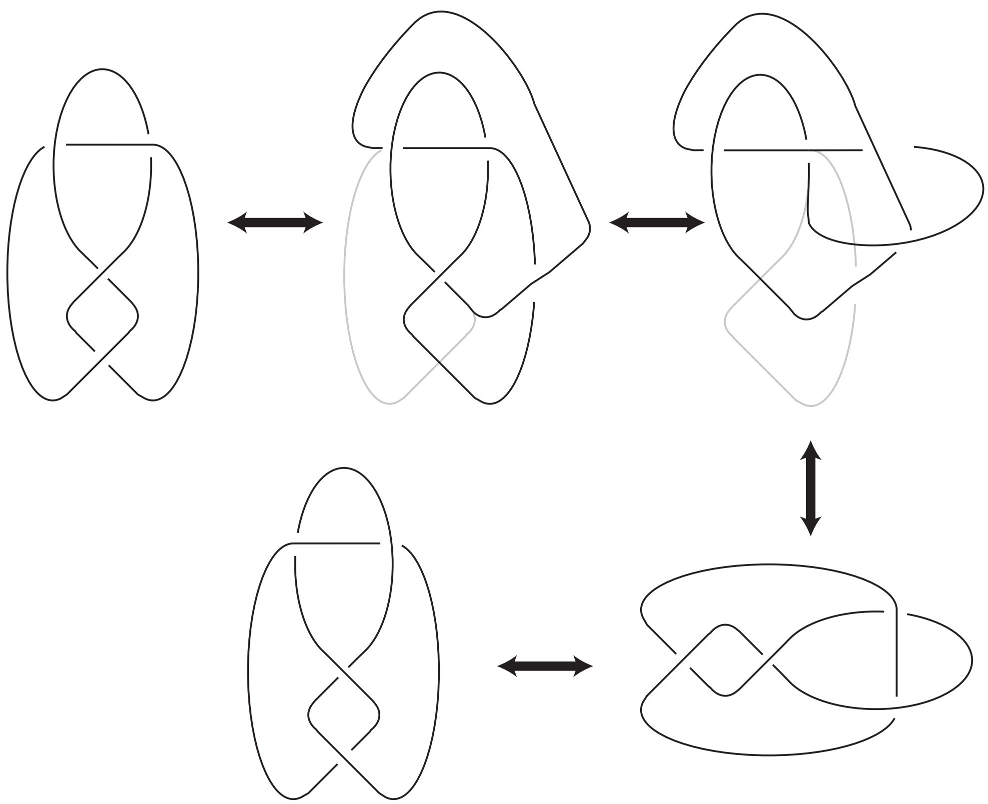


Figure 6: How to get from the figure 8 to its reflection.

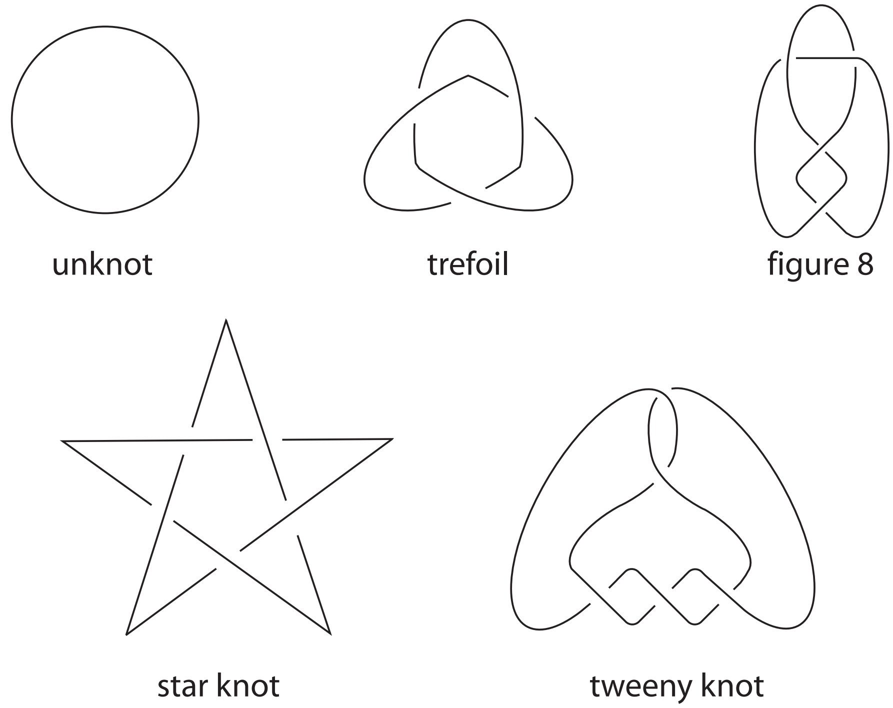


Figure 7: Famous knots in history.

- Write up some definitions (or elicit them from students): A *knot* is a closed curve in space which doesn't intersect itself. A *knot projection* is a picture of the knot with crossings shown by breaks in the curve.
- The knots up to five crossings are famous enough to have their own names. Aside from the three we've already discussed, there are two 5-crossing knots: the star knot and the tweeny knot. To distinguish between these two, label the crossings a, b, c, d, and e. When you trace along the knot, you will pass through each crossing twice before returning to the starting point. For a star knot, the sequence of crossings is repeated twice, i.e., if you visit crossings a, c, d, e, and b in that order to start with, then you will again visit a, c, d, e, and b before returning to the start point.
- It will take some time for students to feel comfortable copying knots into their notes, so consider providing them with handouts of important diagrams. We encourage you to use the names of the knots whenever convenient to assist note takers.

2. Orientation, crossing sign, and linking number. (Reference: §1.4 of [A].)

- We can distinguish between the two trefoils using crossing sign.
- To determine crossing sign, first orient the knot by choosing a direction to travel around the knot (indicated by adding arrows). As in Figure 8, right- and left-handed crossings will be labelled with the "sign" +1 and -1, respectively. There are several, equivalent, ways to determine if a crossing is left or right handed; use whichever works best for you and your students:

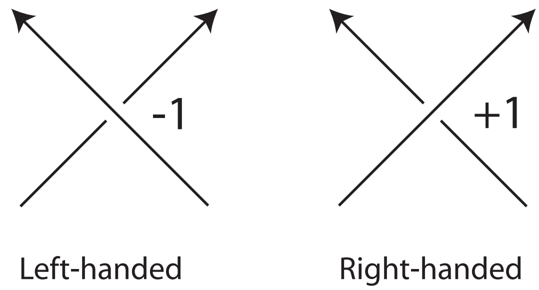


Figure 8: Left- and Right-handed crossings.

Methods for determining crossing sign

For (a) and (b) below, you will need to use your left hand for a left-handed crossing and your right hand for a right-handed crossing. If the knot is drawn on paper, it's often more convenient to move the paper around rather than trying to contort your hand into the correct position.

(a) Place thumb in direction of the overstrand and imagine curling fingers around the overstrand so that fingertips point in the direction of the understrand.

(b) Lay your hand palm down on the paper with fingertips pointing in the direction of the overstrand. Then your thumb will point in the direction of the understrand.

(c) Rotate the understrand so as to make the arrows line up. For a right-handed crossing you will rotate clockwise and for a left-handed crossing you will rotate counter-clockwise. (Compare to rotating the understrand to sweep out water – see notes for Day 2 below – when forming the Lake and Island polynomial).

- A three crossing projection of the right trefoil has all right-hand crossings while a three crossing left trefoil has only left-handed crossings.
- A *link* is a collection of closed curves. The closed curves are called *components* of the link.
- We can often distinguish links using linking number.
- To evaluate the linking number, add the signs of crossings between two components of the link, take the absolute value, and divide by two.
- Some famous links (see Figure 9) are the unlink, Hopf link, King Solomon's link, Whitehead link, and the Borromean rings. Hopf was a German mathematician and J.H.C. Whitehead came up with his link as a counterexample to a project proposed by John Milnor as a way to prove the Poincaré Conjecture. The Borromeo's were an Italian family prominent during the Renaissance and these rings appeared in their family crest. The rings symbolise strength in unity. If any one of the three is removed, the other two fall apart.
- Calculate the linking number of the unlink, Hopf link, and Solomon's link. (In Homework students will learn that the Whitehead link, like the unlink, has linking number zero. So linking number is not a complete invariant, i.e., it's not one to one.)

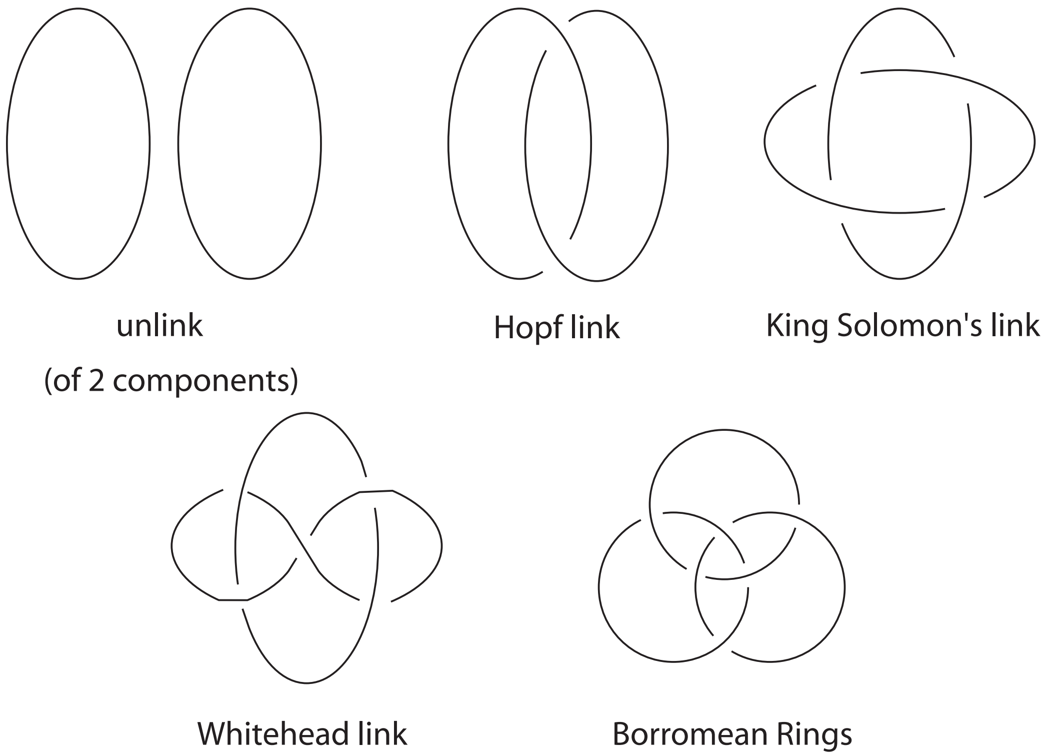


Figure 9: Famous links in history.

Homework:

1. Complete the proof that the trefoil knot has crossing number three by examining projections with two crossings. Try to draw all possible knot projections with two crossings. You should have more than ten projections. Do you think you've drawn them all? Explain why or why not.

   Show that each of the knot projections you've drawn is a projection of the unknot.

2. For each of the following projections (see Figure 10), determine if it represents an unknot, trefoil, figure 8, star knot, or the tweeny knot.

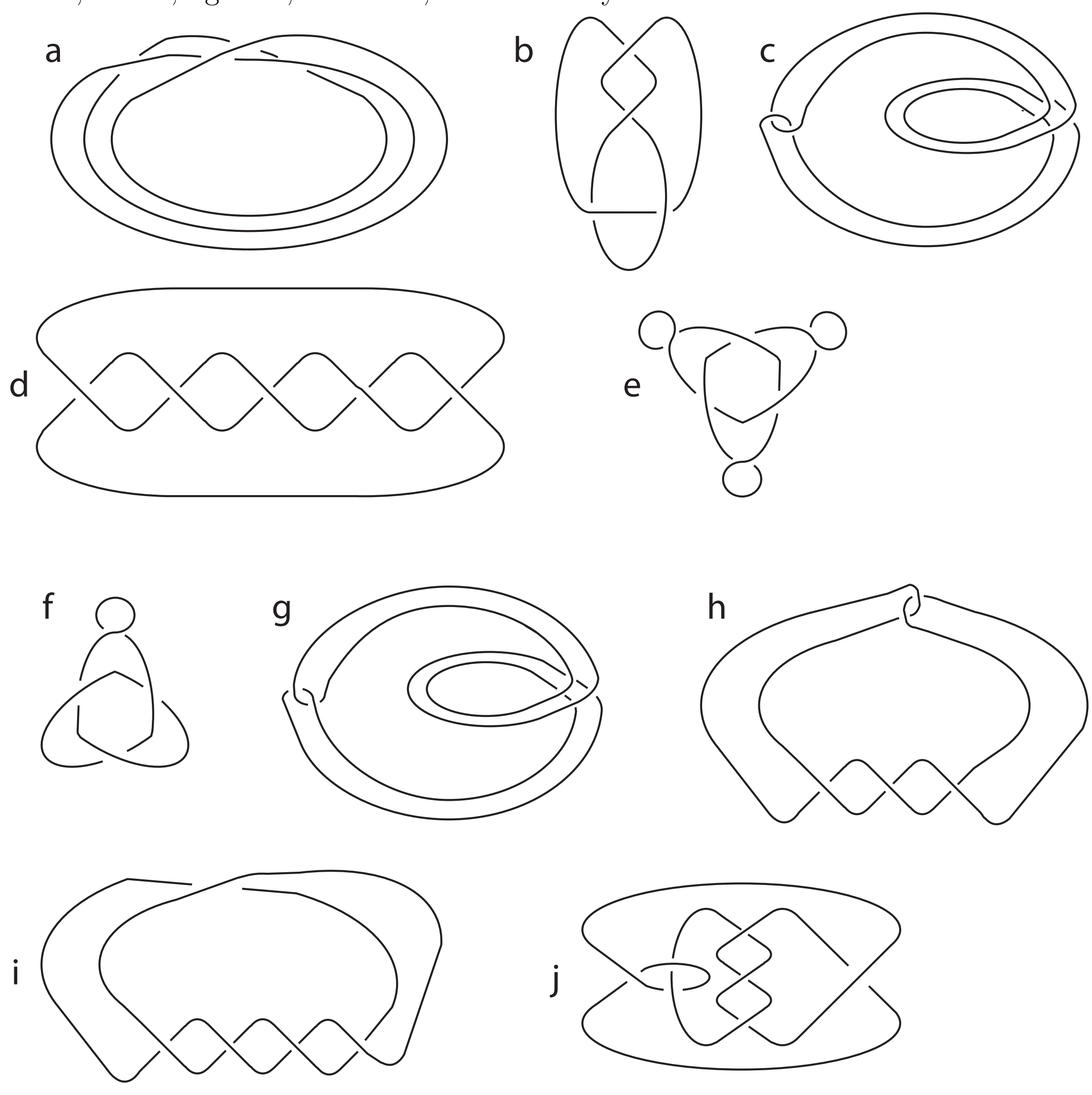


Figure 10: Which knots are these?.

3. Give 5 different projections of the unknot (all different from those in the previous exercise). Give three different projections of the trefoil. Find some "nice" projections of the figure eight knot.

4. Explain why any knot has a projection with over 1000 crossings.

5. An alternating projection is one where the over- and undercrossings alternate as we trace along the curve. Identify the alternating projections in Figure 10. Also list the number of crossings in each of the projections as well as the crossing number of the knot it represents.

Knot	a	b	...
Alt/Non	N	A	...
# Crossings	4	4	...
$C(K)$	0	4	...

   Can you see a connection between crossing number and whether or not a projection is alternating? When does an alternating knot have $C(K)$ crossings? How about a non-alternating knot?

6. Just as there are two trefoils, there are right and left versions of the 5-crossing knots. Draw projections of the left star knot and the right star knot. Which of the two tweeny knots would you call "left", and which "right"?

7. What happens to crossing signs in a knot projection when the orientation is reversed. How about in a link projection? Why do we take the absolute value in computing linking number?

8. Why is the linking number an integer (rather than a half integer)?

9. Calculate the linking number of the links in Figure 11.

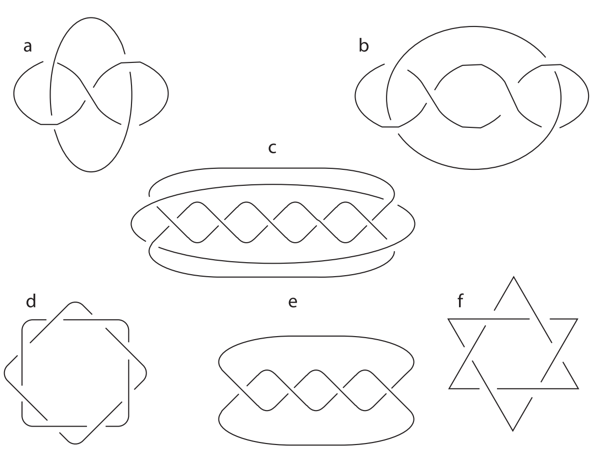


Figure 11: Calculate the linking number.

Extension Problems

1. Show that by changing some undercrossings to overcrossings any projection can be made into a projection of the unknot. (See §3.1 of [A].) (Of course changing crossings will change the knot.)

2. Show that by changing crossings any projection can be made into an alternating projection.

3. A link is Brunnian if the link is not an unlink, but removing any one component results in the unlink. For example, the Borromean rings are Brunnian. Find a Brunnian link of four components. Find Brunnian links with an arbitrary number of components.

4. Find examples of more complicated knots, e.g., knots with 6 or 7 crossings. Which are the same as their mirror reflections?

5. To keep track of knots, mathematicians have organised them into tables. In 1974, an amateur mathematician, K.A. Perko, noticed an error in the knot tables which had gone undetected for almost 100 years. He noticed that two of the following four projections actually represent the same knot. (Knot theorists had thought they were different). Which two are the same knot? (Note: This is a very difficult problem.)

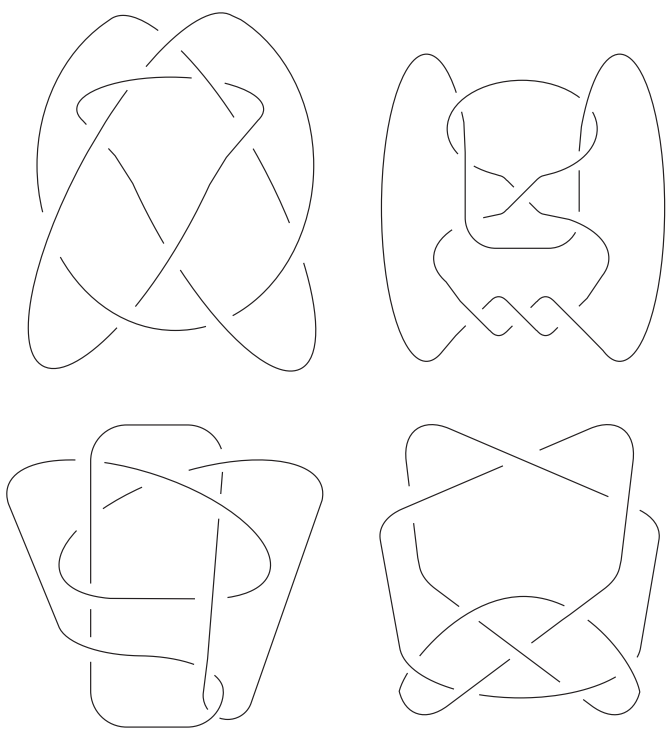


Figure 12: Which two of these four are the same knot?

6. Make up a conjecture and see if you can prove it.

Day 2 — Lake & Island Polynomial and Unknotting number

1. Homework review (Reference: §6.2 of [A].)

   - Revisit Homework including idea of alternating knot and connection with crossing number. Hopefully, students will have made a conjecture. Some may have conjectured that an alternating projection realises the crossing number. This is not quite true. We have to use a *reduced* projection. For example, the following projections of the trefoil knot are alternating, but not reduced. That is why they have more than three crossings.

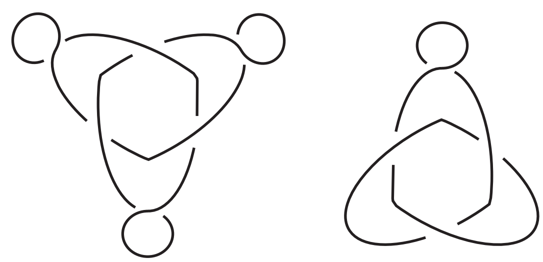


Figure 13: These projections are alternating, but not reduced.

   A reduced projection has no "easily removed" crossings, such as the loops which appear in Figure 13. We can easily remove those loops just by twisting. Figure 14 shows another type of easily removed crossing. Here, one can remove the crossing

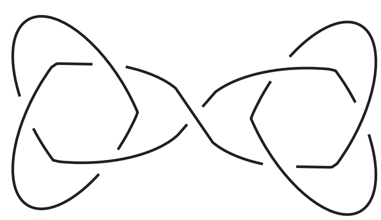


Figure 14: Another type of easily removed crossing.

   at the centre by twisting the whole trefoil on the right (or by twisting the whole trefoil on the left). We say a projection is *reduced* if it does not have "easily removed" crossings of the types illustrated in Figures 13 and 14.

   - The conjecture: "A reduced alternating projection realises crossing number", is one of three conjectures made by the physicist Tait more than a century ago. All three of the conjectures were proved soon after the Jones polynomial was introduced in 1984. In other words, the Jones polynomial radically changed the face of knot theory. Problems that were once very difficult, became relatively easy with this new tool.

- Another conjecture is that a non-alternating projection cannot realise the crossing number. This is true if the knot has an alternating projection. However, some knots have no alternating projection (and, therefore, such a knot must have a non-alternating projection with $C(K)$ crossings). The first example in the knot tables is 8_{19}, shown in Figure 15. The projection shown is reduced, but not alternating.

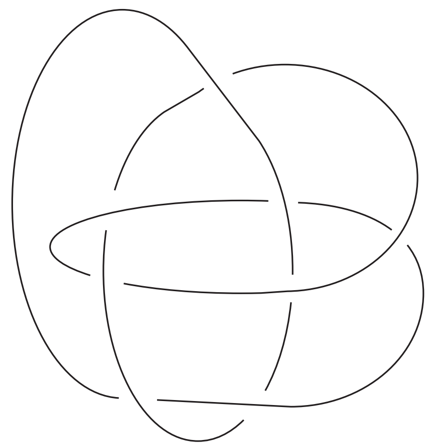


Figure 15: A reduced, non-alternating projection.

- Prompt for other conjectures, observations, or questions arising from homework.

2. Calculate the Lake and Island polynomial of some simple knots and links. (Reference: §6.2 of [A]. The Lake and Island polynomial is usually called the bracket polynomial.)

   - Given an alternating projection, we can shade regions of the projection blue (water) or red (land). (In black and white figures, as well as in the mathlet, we will shade only the water, i.e., land will be "shaded" white.) The convention is that rotating the undercrossing arc clockwise sweeps out water.
   - At each crossing, we can either dig a ditch (connect the two water regions) or build a bridge (connect the land regions). After building bridges and ditches at all crossings, we are left with a number of "circles" or closed loops (since we have no more crossings). The monomial associated with such a *state* (i.e., projection that results from a choice of bridge or ditch at each crossing) is $x^b y^{c-1} z^d$ where b is the number of bridges, c is the number of circles and d is the number of ditches.
   - The Lake and Island polynomial is given by summing these state monomials. For an n-crossing projection, there are 2^n state monomials since, in forming a state, we must choose either a bridge or a ditch at each crossing.
   - Calculate the Lake and Island polynomial of two crossing number one knots (the two leftmost projections of Figure 16).

     The top one yields $x + yz$ and the lower gives $xy + z$.

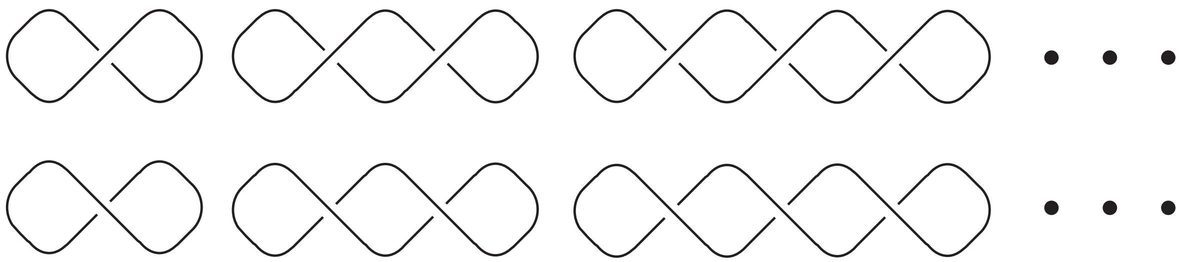


Figure 16: Calculate the Lake and Island polynomials of these projections.

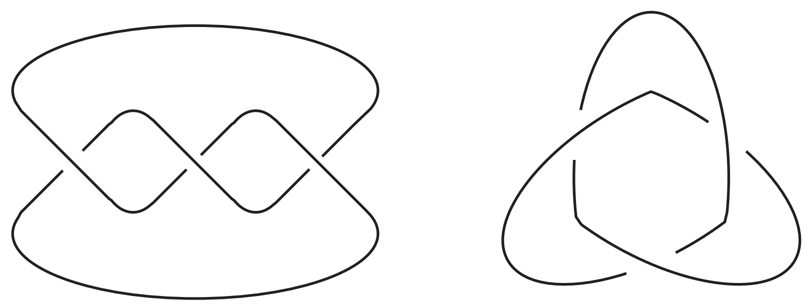


Figure 17: Two projections of the right trefoil.

- Calculate the Lake and Island polynomial of two different projections of the right trefoil. (See Figure 17.)
- Observe that although the two unknots have different polynomials, the trefoils both produce $x^3y^2 + 3xz(xy + z) + yz^3$.
- Ask the students to calculate polynomials of the 2-crossing knots of Figure 16. (Answers: $x^2+2xyz+y^2z^2 = (x+yz)^2$ and $(xy+z)^2$). Present solutions for the 3-crossing knots ($(x+yz)^3$, $(xy+z)^3$) and see if you can get students to conjecture the general pattern for these knots. (This is also included in the homework problems.)
- Note that in each monomial, the x and z exponents sum to the number of crossings. This is a good way to check your work.
- Keep track of student conjectures and discussion. Some of their ideas will appear again in coming classes and homework.

3. Introduce idea of unknotting number via story of Alexander the Great. (Reference: §3.1 of [A]. Barry Cipra's article [C] is also an excellent resource.)

   - We can unknot any knot if we are allowed to make crossing changes. (See Extension 1 from Day 1.) To prove this, take a projection of the knot and as you trace along it, change crossings so that you always cross over a crossing the first time you come to it (and under the second time). This means the string is all piled on top of itself. Looking from the side, we see that the knot is an unknot (see page 59 in §3.1 of [A]).

- The *unknotting number* of a knot K is the least number of crossing changes needed to turn it into the unknot.
- Have the students join hands to form the shape of an unknotting number one knot such as the tweeny knot (see Figure 18). Then get them to try to untangle

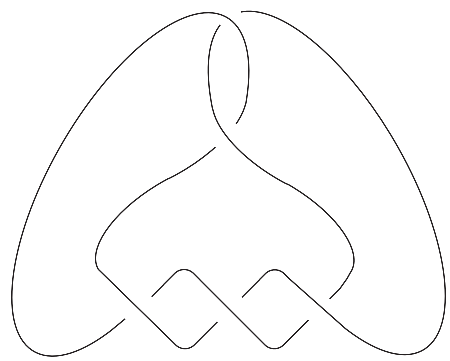


Figure 18: The tweeny knot.

(without letting go of their hands). After they've struggled for a while, change a crossing and let them struggle some more. After a few crossing changes, they'll be undone.

- Have the students form the tweeny knot again. This time make only one (the correct) crossing change and watch them unravel. (For example, changing either of the two crossings at the top of Figure 18 would undo the knot.)
- The point is, this is an unknotting number one knot, but there are other ways to unknot it. It's not so easy to find the best way.
- To show a knot has unknotting number one is straightforward — just find the good crossing change.
- To show a knot has unknotting number two or greater is hard. You must demonstrate that there's no way to unknot with only one crossing change.
- Example of $(5, 1, 4)$ pretzel knot: Have students investigate the unknotting number of the knot using the two projections of Figure 19. (This is also a homework problem.) Surprisingly, although three crossing changes are needed to unknot the knot in the 10 crossing projection at left (namely, three of the five on the left side of the knot), only two are needed in the other, apparently more complicated, projection at right (namely, the one at the centre of the knot and one of the two new crossings). This knot has crossing number 10 (the 10 crossing projection is reduced and alternating) and unknotting number 2. However, as was first proved by Steve Bleiler and Yasutaka Nakanishi (independently), the unknotting number

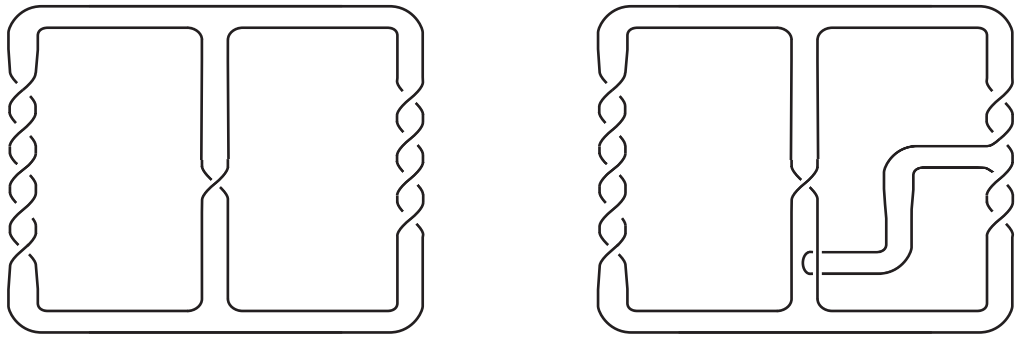


Figure 19: The $(5, 1, 4)$ pretzel knot.

is not realised by any 10 crossing projection. In other words, to find the unknotting number, it's not enough to simply look at the "simplest" projections of a knot. In some cases, it's necessary to consider more complicated projections.

- The unknotting number is often called the Gordian number in honour of Alexander the Great's solution to the Gordian knot.

Homework

1. Use the mathlet to work out the Lake & Island polynomials of the eight given projections.

2. What are the Lake and Island polynomials of the projections in Figure 20?

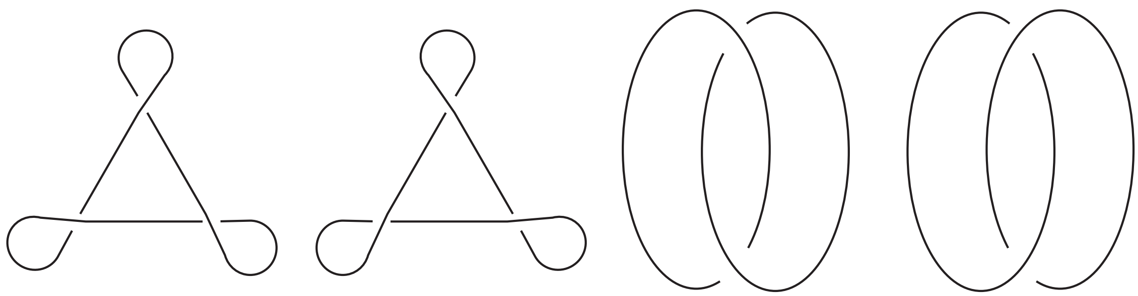


Figure 20: What are the Lake and Island polynomials?

3. Draw a projection and choice of bridges and ditches that result in xy^3z^2.

4. Draw a projection and choice of bridges and ditches that result in x^ay^2z where $a = 1, 2, 3, \ldots$. How about $a = 0$?

5. Find the Lake and Island polynomial for the following projections with 1,2, and 3 crossings. (See Figure 21.) What do you think is the polynomial for the corresponding projection with n crossings? (Hint: It involves a binomial.)

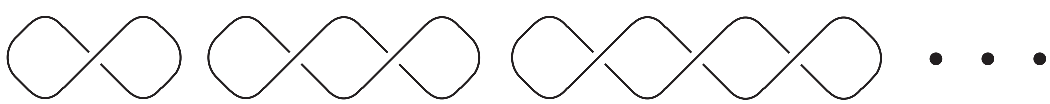


Figure 21: What is the polynomial for n crossings?

6. Show that the trefoil and figure eight knots have unknotting number 1.

7. Show twist knots have unknotting number one. (See Figure 22.)

8. Which of the knots up to six crossings have unknotting number one? (The famous knots are all knots up to five crossings. In addition, there are three 6-crossings knots as shown in Figure 23.)

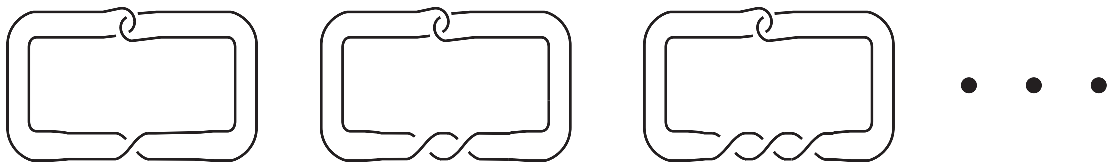


Figure 22: The twist knots.

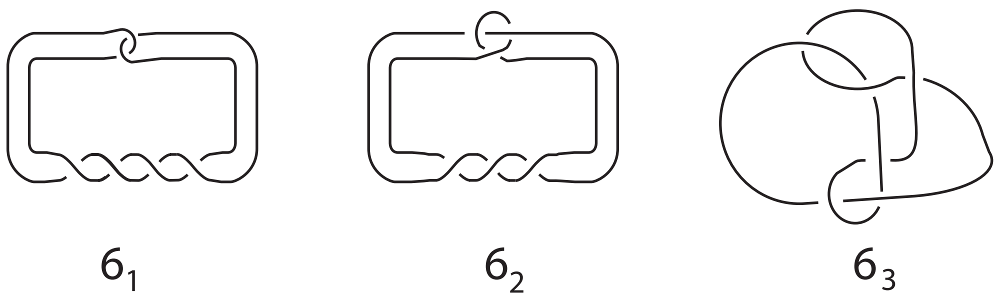


Figure 23: The 6-crossing knots.

9. Find the unknotting number of the knots 3_1, 5_1, 7_1, ... (see Figure 24). Make a conjecture about the unknotting number of knots in this sequence. Show that your guess is an upper bound for the unknotting number.

10. Figure 19 shows two projections of the $(5, 1, 4)$ pretzel knot. The projection at left can be unknotted with 3 crossing changes. Find those 3 (there's more than one set of 3 that will work). The projection at right requires only 2 crossing changes. Find them.

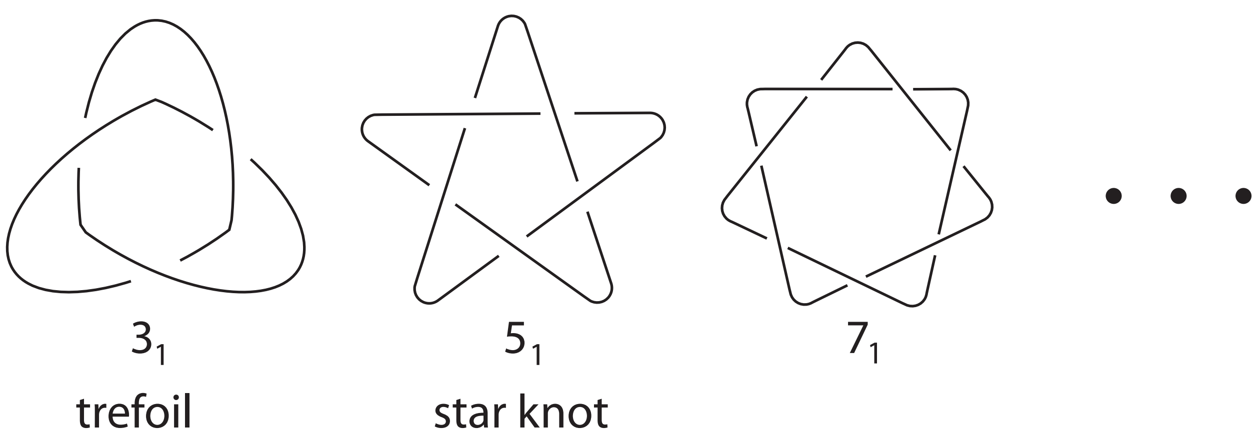


Figure 24: What is the unknotting number?

Extensions

1. Find the Lake and Island polynomial of the projection shown in Figure 25 (Hint: Work locally).

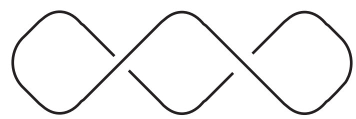


Figure 25: What is the Lake and Island polynomial?

2. Can the y exponent of the Lake and Island polynomial be bounded in terms of the crossing number of the projection? (The x and z exponents are both at most equal to the crossing number.)

3. Find an inequality relating the crossing number of a knot and the unknotting number (Hint: See Extension 1 from Day 1).

4. In 1993, Kronheimer and Mrowka [KM] proved a conjecture of Milnor: The unknotting number of a (p, q) torus knot is $(p - 1)(q - 1)/2$. Torus knots are discussed in Section 5.1 of Colin Adams's *The Knot Book*[A] and we already know some examples. The knots of Figure 24 are the $(p, 2)$ torus knots: the trefoil is a $(3, 2)$ torus; the star knot is a $(5, 2)$ torus; the knot 7_1 (see Figure 24) is a $(7, 2)$ torus; etc.

   Demonstrate that the $(4, 3)$ torus knot is 8_{19} in the knot tables (see Figure 15). Show that it has unknotting number at most $(4 - 1)(3 - 1)/2 = 3$ by finding three crossing changes which unknot.

5. Show that the knots 7_2, 7_6, and 7_7 have unknotting number one. Show that the knots 7_3, 7_4, and 7_5 have unknotting number at most 2. (The other seven crossing knot, 7_1 is a torus knot. See the previous exercise for its unknotting number.)

Day 3 — Reidemeister Moves

1. Invariants (References: Invariant is defined in §1.4 of [A]. Complete invariant is discussed in §6.3.)

   - Crossing number, linking number, and unknotting number are examples of *knot* (or link) *invariants*.
   - For example, Crossing #(unknot) = C(unknot) = 0, Crossing #(trefoil) = C(trefoil) = 3, Crossing #(figure eight) = C(figure eight) = 4. For unknotting number, we have Unknotting#(unknot) = U(unknot) = 0, Unknotting#(trefoil) = U(trefoil) = 1, and Unknotting#(figure eight) = U(figure eight) = 1. (These last two were homework.) As the notation suggests, a knot invariant can be thought of as a function on the set of knots.
   - Both of these invariants allow us to distinguish the unknot from all other knots. However, neither of these represent a one-to-one function. In particular, in analogy with $f(x) = x^2$, both of these give the same value to the right and left trefoil. An invariant which represents a one-to-one function is called a *complete invariant*. So far, mathematicians know of no complete invariants. It is a big research problem to find one. (To be precise, there are no known complete *numeric* invariants, i.e., functions into the set of real numbers or into the set of sequences of real numbers. The knot group, together with information about its subgroups, is an example of a complete invariant, but it's almost as hard to work with as the knot itself.)
   - Crossing number and unknotting number are quite difficult to determine, especially for knots with crossing number > 3 and knots with unknotting number > 1. For crossing number we know that reduced alternating projections realise the crossing number. For unknotting number, recent (1993) work of Kronheimer and Mrowka shows that unknotting number of a (p,q) torus knot is $(p-1)(q-1)/2$. (See Extension 4 from yesterday's notes.)
   - Crossing number and unknotting numbers are invariants by definition. They don't depend on the projection as they are defined to be the minimum over all projections. The very thing which makes them difficult to calculate also ensures that they are invariants.
   - On the other hand, linking number is quite easy to compute, but how do we know it's an invariant? This requires a proof.

2. Introduce Reidemeister moves and use them to show linking number is a link invariant. (Reference: §§1.3 & 1.4 of [A].)

   - Reidemeister was a German mathematician who wrote the first knot theory text book.
   - Introduce the Reidemeister moves as three things you can do to a knot diagram which will obviously not change the knot. (See Figure 26.)

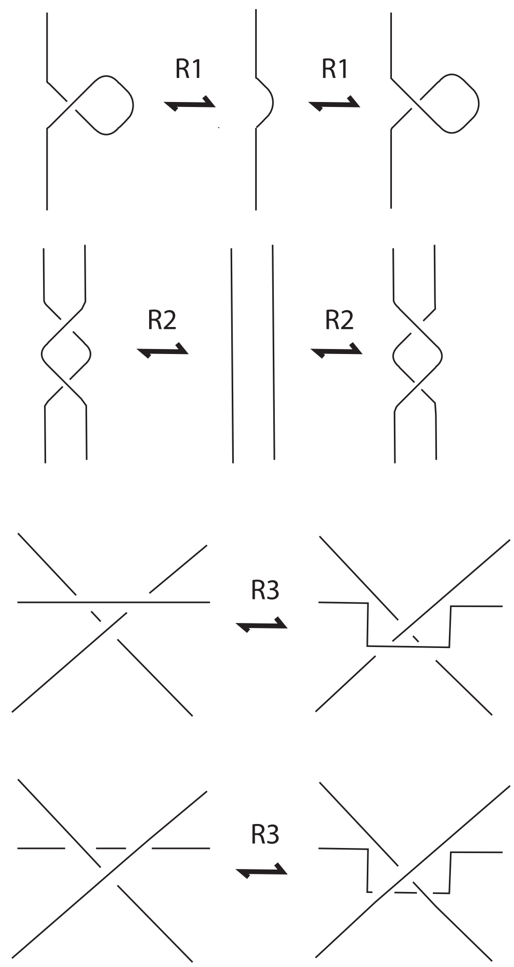


Figure 26: The Reidemeister moves.

- We will denote the three types of Reidemeister moves as R1, R2, and R3 (sometimes called "twist", "poke", and "slide").
- Theorem: Two knots are equivalent if and only if they have Reidemeister equivalent projections.

  Two projections are Reidemeister equivalent if one can be obtained from the other by a sequence of Reidemeister moves. One direction of the theorem should be clear; when we make a Reidemeister move, we will not change the knot.

  The power of the theorem lies in the converse. It allows us to focus on the three moves; any other legitimate manipulation of our knot (i.e., not involving cutting the string), no matter how complicated, can be reduced to a sequence of Reidemeister moves. (See Extension 3 for an outline of the proof of this theorem).
- Show how Reidemeister moves can be used to change one representation of the trefoil to the other. (See Figure 27. Note that in this figure, the arrows that are not labelled represent moves which merely "straighten lines" without changing

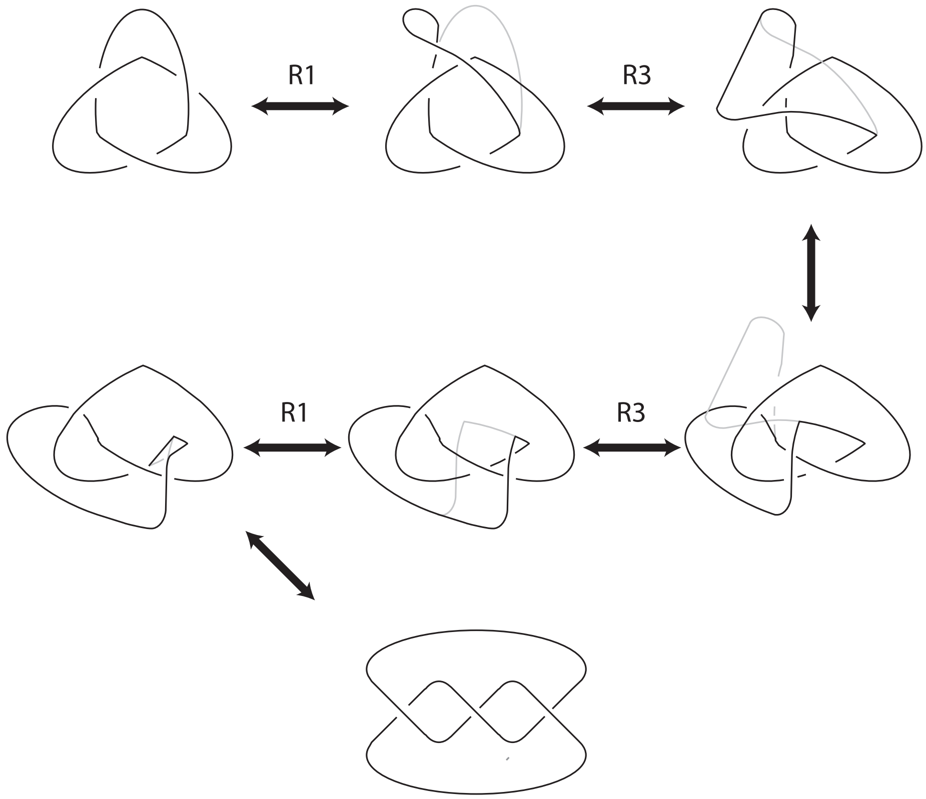


Figure 27: The two projections of the trefoil are Reidemeister equivalent.

any crossings. Such a straightening move is sometimes called an “R0” move.)

- Have students explore using Reidemeister moves to move between different projections of the same knot (for example, different projections of the unknot).
- Although Reidemeister moves are very useful as a theoretical tool, we ordinarily do not use them to show two projections are equivalent as that is quite tedious and involves adding a lot of extra steps. For example, compare Figures 2 and 27.
- Show that linking number is not changed by R2 moves. As illustrated in Figure 28, whichever R2 move we apply, that portion of the projection shown in the figure contributes a total of zero to the sum of the crossing signs. This is also true if the crossings shown represent self-crossings, or if the orientation of one or the other arc is changed. Since R2 moves will not change the sum of the crossing signs, the linking number is not affected by an R2 move.
- Have students investigate the effect of R1 moves on linking number. (The R3 move is left as a homework exercise.)
- Emphasise that, by showing the linking number does not change under the three Reidemeister moves, we’ve shown that linking number is a link invariant. In other words, different projections of the same link will have the same linking number.
- Revisit Whitehead link to illustrate that linking number doesn’t allow us to tell all links apart. The Whitehead link and unlink both have linking number zero

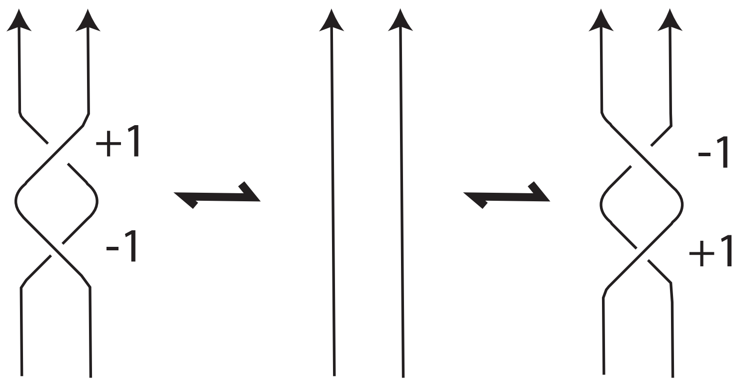


Figure 28: R2 moves do not affect linking number.

although they are different links. (In other words, linking number is not a one-to-one function. It is not a complete invariant.)

3. The Lake and Island polynomial of non-alternating projections

   - Give a non-alternating diagram (such as Figure 29) and ask students to shade land and water.

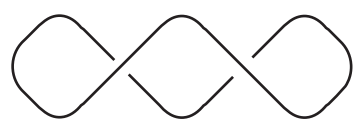


Figure 29: Shade the land and water.

   - Students will realise that it's not possible to consistently shade the regions of a non-alternating projection.
   - However, we can still calculate a Lake and Island polynomial by thinking of bridges and ditches as being constructions local to the crossing.

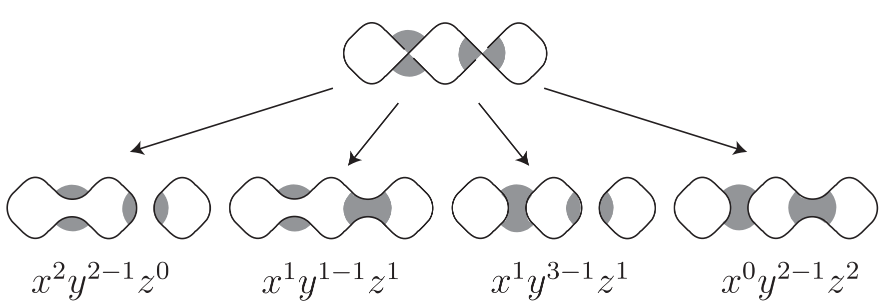
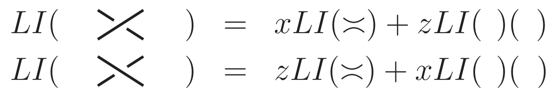


Figure 30: The Lake and Island polynomial is $x^2y + xz + xy^2z + yz^2$

- This idea of local constructions is captured by the equations



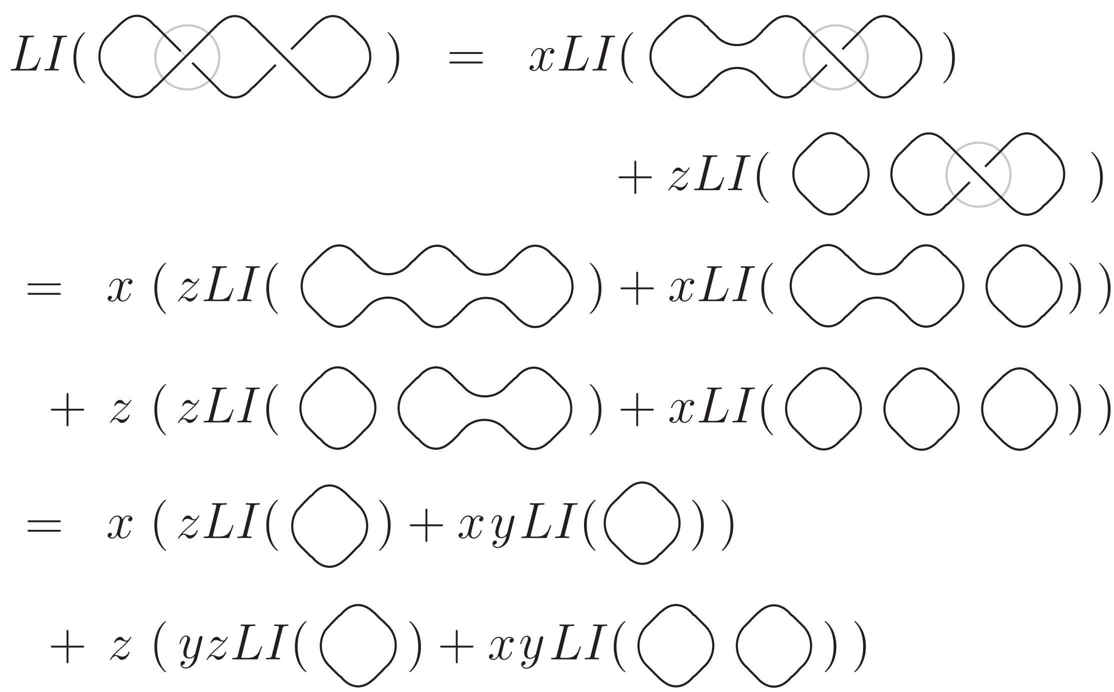
In these equations, we've abbreviated the drawing of a knot projection by focusing on the neighbourhood of a crossing. The Lake and Island polynomial with a crossing as at left, can be replaced with x and z times the polynomials of projections where the crossing has been replaced by a bridge or ditch.

- Through repeated application of these equations, we eventually eliminate all the crossings in our projections leaving us with a collection of unknots. The calculation is completed using two additional rules.

$$LI(L \cup \bigcirc) = yLI(L)$$
$$LI(\bigcirc) = 1$$

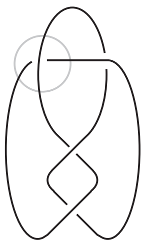
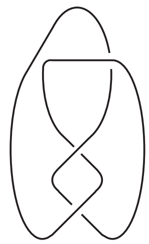
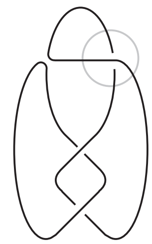
In words, the polynomial of a link L together with a disjoint unknot is y times the polynomial of L. The polynomial of the unknot is 1.

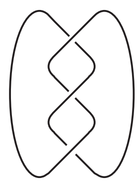
- Let's repeat the calculation above using these four equations.

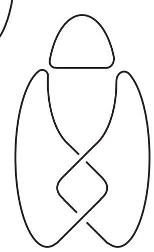
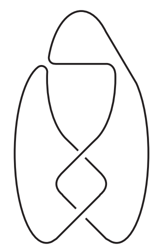


$$= x(z1 + xy1) + z(yz1 + xyy1)$$
$$= xz + x^2y + yz^2 + xy^2z$$

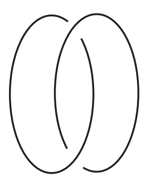
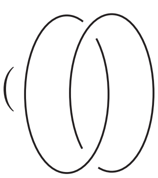
- Using the four equations, we can determine the Lake and Island polynomial of any projection, alternating or not. In addition, the recursion in these equations means that we can build on earlier knowledge to determine the polynomials of more complicated projections. For example, we can use the polynomials of the trefoil and Hopf link to determine that of the figure 8 knot.

$LI($  $)$ $=$ $xLI($  $) + zLI($  $)$

$=$ $xLI($  $)$

$+ z(zLI($  $) + xLI($  $))$

$$= x(x^3y^2 + 3x^2yz + 3xz^2 + yz^3)$$

$+ z(yzLI($  $) + xLI($  $))$

$$= x^4y^2 + 3x^3yz + 3x^2z^2 + xyz^3)$$

$$+ z(yz + x)(x^2y + 2xz + yz^2)$$

$$= x^4y^2 + 4x^3yz + 5x^2z^2 + 4xyz^3 + x^2y^2z^2 + y^2z^4$$

Homework

1. (a) Draw a sequence of projections showing that projection below represents the unknot.

   (b) Explain why any Reidemeister move you make on this projection will increase the number of crossings.

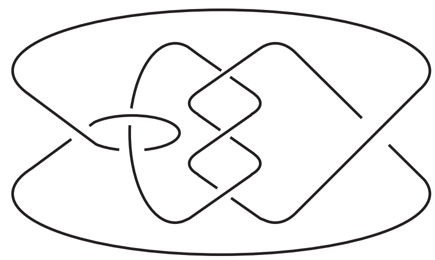


Figure 31: A nasty unknot.

2. Complete the proof that linking number is a link invariant by showing it doesn't change under a type 3 Reidemeister move.

3. Find the Lake and Island polynomials of the projections in Figure 32.

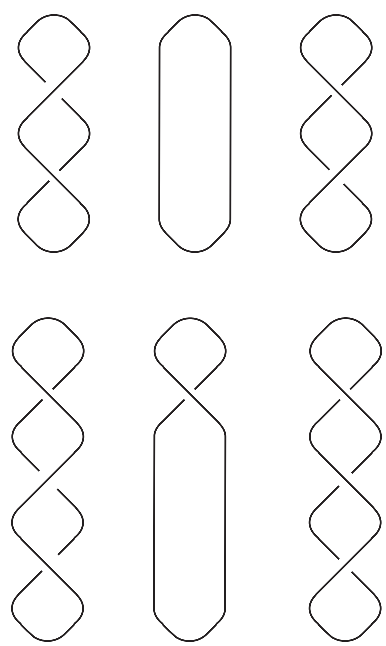


Figure 32: Projections which differ by an R2 move.

4. Use the four equations introduced in class today to calculate the Lake and Island polynomials of the projections in Figure 33. (You may already have calculated some of these in earlier homework. Please verify your answer using the new method.)

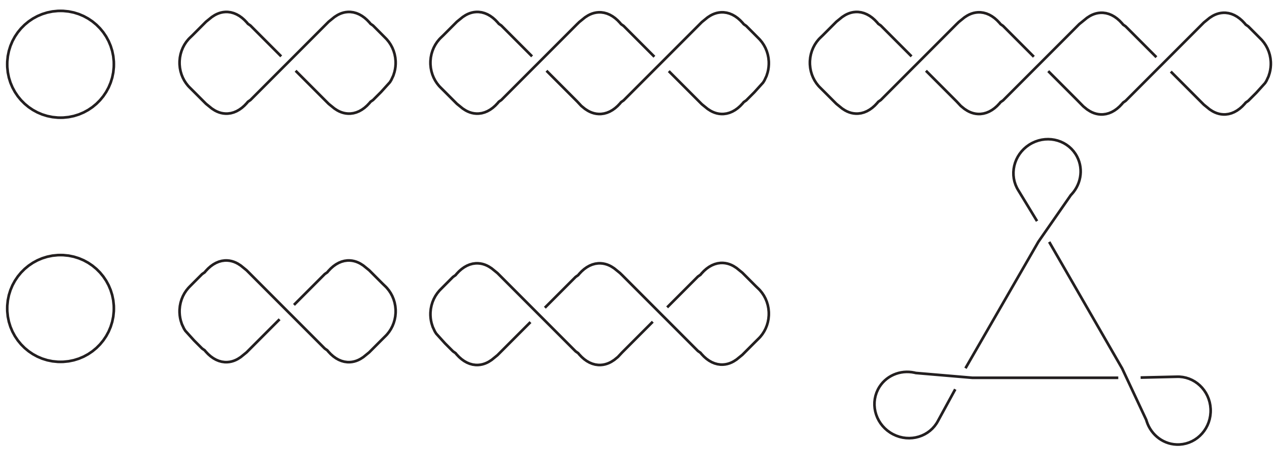


Figure 33: Projections which differ by an R1 move.

5. The *writhe* of a projection is the sum of the crossing signs. For example, the usual projections of the right trefoil have writhe 3. How does the writhe change if one changes the knot's orientation? Calculate the writhes of the projections in Figure 33. Is there a connection between the writhe and the polynomial of these projections?

6. Is the Lake and Island polynomial a function? What is the domain and what is the range? Is it one-to-one?

Extensions

1. Use the Reidemeister moves to show that tricolourability is a knot invariant. (See §1.5 of Colin Adam's *The Knot Book*[A] for a definition of tricourability.) What are the domain and range of the tricolourability function? Is it one-to-one? Is it onto?

2. Which of the famous knots are three-colourable. Which are five-colourable?

3. How could you show that the three Reidemeister moves are enough? One approach is to look at Delta moves. (This idea is explained in more detail in the book *Knots* by Burde and Zieschang [BZ].) First, note that we can approximate knots as accurately as we like by polygonal paths. That is, instead of curves, we can use straight line segments. We can change such a polygonal path by a Delta move. For such a move, take a triangle (in a plane) whose interior does not intersect the knot. The Delta move involves replacing one edge of the triangle with the other two (or vice versa). That is, if one edge is part of a polygonal path defining a knot, then replacing the edge by the other two edges of the triangle is called a Delta move. Now, there are two things to show:

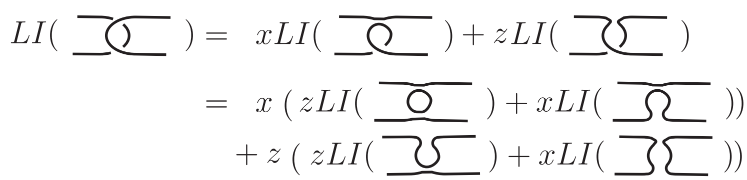
   (a) Two knots K_1 and K_2 are equivalent iff they are Delta-equivalent (i.e., equivalent by Delta moves).

   (b) Two knots K_1 and K_2 are Delta equivalent iff they are Reidemeister equivalent.

Day 4 — Jones polynomial.

1. Turning the Lake & Island polynomial into an invariant: First eliminate y and z in favour of x. (Reference: §6.1 of [A].)

   - As it stands, the Lake and Island polynomial is not a knot invariant.
   - For linking number, we were able to use the Reidemeister moves to see that it is an invariant. Here, we will use the Reidemeister moves to investigate how far from an invariant the Lake and Island polynomial is.
   - We begin with R2. Note that we can relate the polynomials before and after an R2 move:



$$
\begin{aligned}
&= x(yzLI(\asymp) + xLI(\asymp)) + z(zLI(\asymp) + xLI(\,)(\,))\\
&= (x^2 + xyz + z^2)LI(\asymp) + xzLI(\,)(\,)
\end{aligned}
$$

   - If the Lake and Island polynomial is to be an invariant, the right-hand side must be simply $LI(\,)(\,)$. This is a conundrum, but the solution is a common trick in mathematics: we define it to be so and deal with the consequences.
   - To make the right hand side $LI(\,)(\,)$ requires that $xz = 1$, so we can eliminate z in favour of x: $z = x^{-1}$.
   - Similarly, we need $x^2 + xyz + z^2 = 0$. In light of our assigning $x = z^{-1}$, this gives us a way to write y in terms of x too:

$$
\begin{aligned}
& x^2 + xyz + z^2 = 0\\
\Rightarrow\ & x^2 + y + x^{-2} = 0\\
\Rightarrow\ & y = -(x^2 + x^{-2})
\end{aligned}
$$

   - In other words, by setting $z = x^{-1}$ and $y = -(x^2 + x^{-2})$ the Lake and Island polynomial becomes invariant under R2 moves. Have the students verify that these substitutions into the examples from homework will make the polynomials equal for projections which differ only by an R2 move.
   - In fact, this choice of y and z will also verify R3. (Homework exercise).
   - Note that on making these substitutions, the polynomials become "Laurent" polynomials, i.e., the x exponents may be negative integers.

- Our four equations can now be written entirely in terms of x:

$LI($  $) = xLI(\asymp) + 1/xLI(\ )(\ )$

$LI($  $) = 1/xLI(\asymp) + xLI(\ )(\ )$

$$LI(L \cup \bigcirc) = (-x^2 - x^{-2})LI(L)$$

$$LI(\bigcirc) = 1$$

Advise students to use this form of the equations from now on.

2. Introduce writhe to deal with R1.

- Review the polynomials of projections differing by an R1 move calculated in homework. Note that after an R1 move, the polynomials changes by a factor of $-x^{\pm 3}$. (Actually, in the homework, the students probably noticed that R1 introduces $xy + z$ or $x + yz$. This is a good opportunity to derive the useful equations $xy + z = -x^3$ and $x + yz = -x^{-3}$.)
- The results can be summarised by the following equations.

$LI($  $) = -x^3 LI(\text{———})$

$LI($  $) = -x^{-3} LI(\text{———})$

- Note that R2 and R3 moves do not change the writhe of a projection. This is a homework exercise and is essentially the same argument as was used to show R2 and R3 don't change linking number. (See Figure 28 from Day 3's notes.) Since R1 changes the writhe by one and changes the Lake and Island polynomial by a factor of $-x^3$, we can balance these two effects by multiplying the Lake and Island polynomial by $(-x^3)^w$ where w denotes the writhe.
- In other words, the Laurent polynomial

$$V(L) = (-x^3)^{w(L)} LI(L)$$

is an invariant of oriented links. It is called the Jones polynomial. The letter V is in honour of Vaughn Jones who discovered this invariant.
- Verify that the Jones polynomial is an invariant by calculating the Jones polynomial of several different projections of the same knot (for example, the unknots of Figure 33).
- Note that, for a knot, the writhe doesn't depend on orientation, so that $V(K)$ is an invariant of knots (not just oriented knots).

3. The Jones polynomial is a good invariant.

- The Jones polynomial is a good invariant in that it is excellent at distinguishing knots and relatively easy to calculate. That is, there is an algorithm to calculate the Jones polynomial, which, in principle, can be carried out for *any* knot we care to investigate. This is very different from the situation for crossing and unknotting number where we can easily calculate the invariant only for special kinds of knots (e.g., reduced alternating knots or torus knots).
- As an example of its power at distinguishing knots, note that the Jones polynomial differentiates the left and right trefoils. Let's calculate the Jones polynomial of the right trefoil. As we've seen (notes from Day 2), the Lake and Island polynomial is

$$\begin{aligned} x^3y^2 + 3xz(xy+z) + yz^3 &= x^3(-x^2 - x^{-2})^2 + 3x1/x(-x^3) + (-x^2 - x^{-2})(1/x)^3 \\ &= x^3(x^4 + 2 + x^{-4}) - 3x^3 - x^{-1} - x^{-5} \\ &= x^7 - x^3 - x^{-5} \end{aligned}$$

  Since the writhe is 3, the Jones polynomial is

$$V(\text{R. trefoil}) = -(x^3)^3(x^7 - x^3 - x^{-5}) = -x^{16} + x^{12} + x^4.$$

- Have students verify that the Jones polynomial of the left trefoil is $-x^{-16} + x^{-12} + x^{-4}$. Thus, the Jones polynomials of the left and right trefoils are different.
- In general, if K and K^* are mirror reflections, then $V(K^*)$ and $V(K)$ differ by interchanging x and $1/x$. ("all the exponents change sign"). Essentially, this is because a mirror reflection interchanges bridges and ditches. So, it swaps 'x's with 'z's. Since $z = x^{-1}$, this amounts to replacing x with its reciprocal. For example, the Jones polynomial of the figure eight knot is $x^8 - x^4 + 1 - x^{-4} + x^{-8}$ so that $V(x) = V(x^{-1})$. This corresponds to the fact that the figure eight knot is the same as its mirror reflection.
- The Jones polynomial differentiates the two trefoils, something none of our other invariants were able to do. It also yields a different polynomial for each knot up to 10 crossings (about 250 knots).
- There are examples of different knots, the simplest examples having crossing number 11 (see Figure 34), which have the same Jones polynomial. So, the Jones polynomial is not a complete invariant. However, it's still not known whether or not the unknot is the only knot with Jones polynomial 1. Indeed, this is a big research question. In 2002, Thistlethwaite[T] found a two-component link with 15 crossings which has Jones polynomial equal to that of the unlink of two components. There are still many new discoveries waiting to be made in this field.

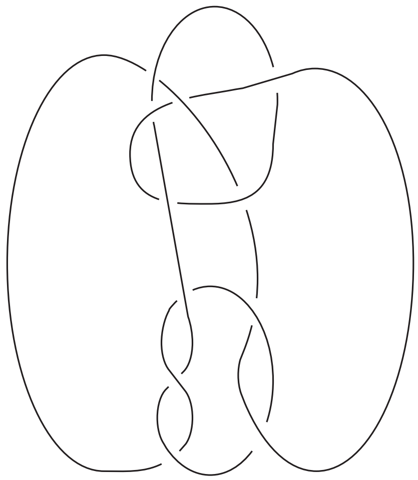


Figure 34: Two different knots that have the same Jones polynomial.

Homework

1. Prove that the five famous knots are all different from one another by verifying that their Jones polynomials are as follows:

$$
\begin{aligned}
V(\text{ L. trefoil }) &= -x^{-16} + x^{-12} + x^{-4} \\
V(\text{ R. trefoil }) &= -x^{16} + x^{12} + x^{4} \\
V(\text{ figure 8 }) &= x^{8} - x^{4} + 1 - x^{-4} + x^{-8} \\
V(\text{ star knot }) &= -x^{28} + x^{24} - x^{20} + x^{16} + x^{8} \\
V(\text{ tweeny }) &= -x^{24} + x^{20} - x^{16} + 2x^{12} - x^{8} + x^{4}
\end{aligned}
$$

2. Actually, you may have got a slightly different answer than what is written above for the two 5-crossing knots. Discuss the possible Jones polynomials for 5-crossing knots. (Hint: Why do we get two different answers for the trefoil?)

3. Complete the proof that the Jones polynomial is a knot invariant by showing:

   (a) If we make the substitution $z = 1/x$ and $y = -(x^2 + x^{-2})$, then the Lake and Island polynomial is invariant under R3.

   (b) Writhe does not change under R2 or R3.

4. Is writhe a knot invariant?

5. What is the Jones polynomial of the trivial link of n components?

6. Show that the Whitehead link is different from the unlink of two components by calculating its Jones polynomial. (Recall that linking number did not distinguish between these two.)

Extensions

1. Show that span$(LI(K))$, the difference between the highest and lowest x degree amongst the monomials in the Lake and Island polynomial $LI(K)$, is a knot invariant. (See §6.2 of *The Knot Book* by Colin Adams [A]. Another good reference is §2.3 of *Knots and Surfaces* by Gilbert and Porter[GP].) For example, the Lake and Island polynomial of the left trefoil is $-x^5 - x^{-3} + x^{-7}$ so that the span is $5 - (-7) = 12$. In fact, one can show that the span is at most $4c$ where c is the number of crossings in the projection used to calculate $LI(K)$. Moreover, for a reduced alternating projection, the span is exactly $4c$. Since the span doesn't depend on the projection (i.e., the span is a knot invariant), this shows that any reduced alternating projection of the knot realises the crossing number.

   This proves the first of the three Tait conjectures. The second Tait conjecture states that every reduced, alternating projection of a knot has the same writhe. (For example, the two standard drawings of a right trefoil both have writhe 3.) The third asserts that one can move from any reduced, alternating projection to any other through a sequence of "flype" moves (See Section 6.2 of *The Knot Book* [A]). These conjectures were made about 100 years ago. Although many great mathematicians tried to prove them, they stood the test of time. However, once the Jones polynomial was discovered, all three of the conjectures were proved in the space of a few years. This is one measure of the profound effect that the discovery of the Jones polynomial has had on this field of research.

2. The trip matrix [Z], T, is an $n \times n$ matrix, whose entries are 0's and 1's. It's derived from an n-crossing projection of the knot K and determines the Jones polynomial of K as follows:

$$V(K) = (-x^3)^{w(K)} \sum_D x^{-\text{null}(D)} x^{\text{rank}(D)} (-x^2 - x^{-2})^{\text{null}(T+D)}.$$

   where the sum is over diagonal matrices D with 0,1 entries and "null" and "rank" are the nullity and rank of the matrices over the field of integers mod 2. Verify that using the trefoil trip matrix

$$T = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

   in this formula does indeed give the Jones polynomial of the trefoil knot. Is this a left or a right trefoil?

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