

February 8: Ch 2, 5

Note Title

2/7/2010

2D: Mapping class group of the torus:

Let $*$: $\text{Aut}(T^2) \rightarrow \text{GL}(2, \mathbf{Z})$, $*(h) = h_*$ where if $h(L) = aL + bM$ and if $h(M) = cL + dM$, then

$$h_* = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Questions: why does $\det(h_*) = \pm 1$ (See Ex 7)why is $*$ a homomorphism?*composition = matrix multiplication*

$$\begin{aligned} & \left| \begin{matrix} (aL+bM) \wedge (cL+dM) \\ = 1 \end{matrix} \right| \end{aligned}$$

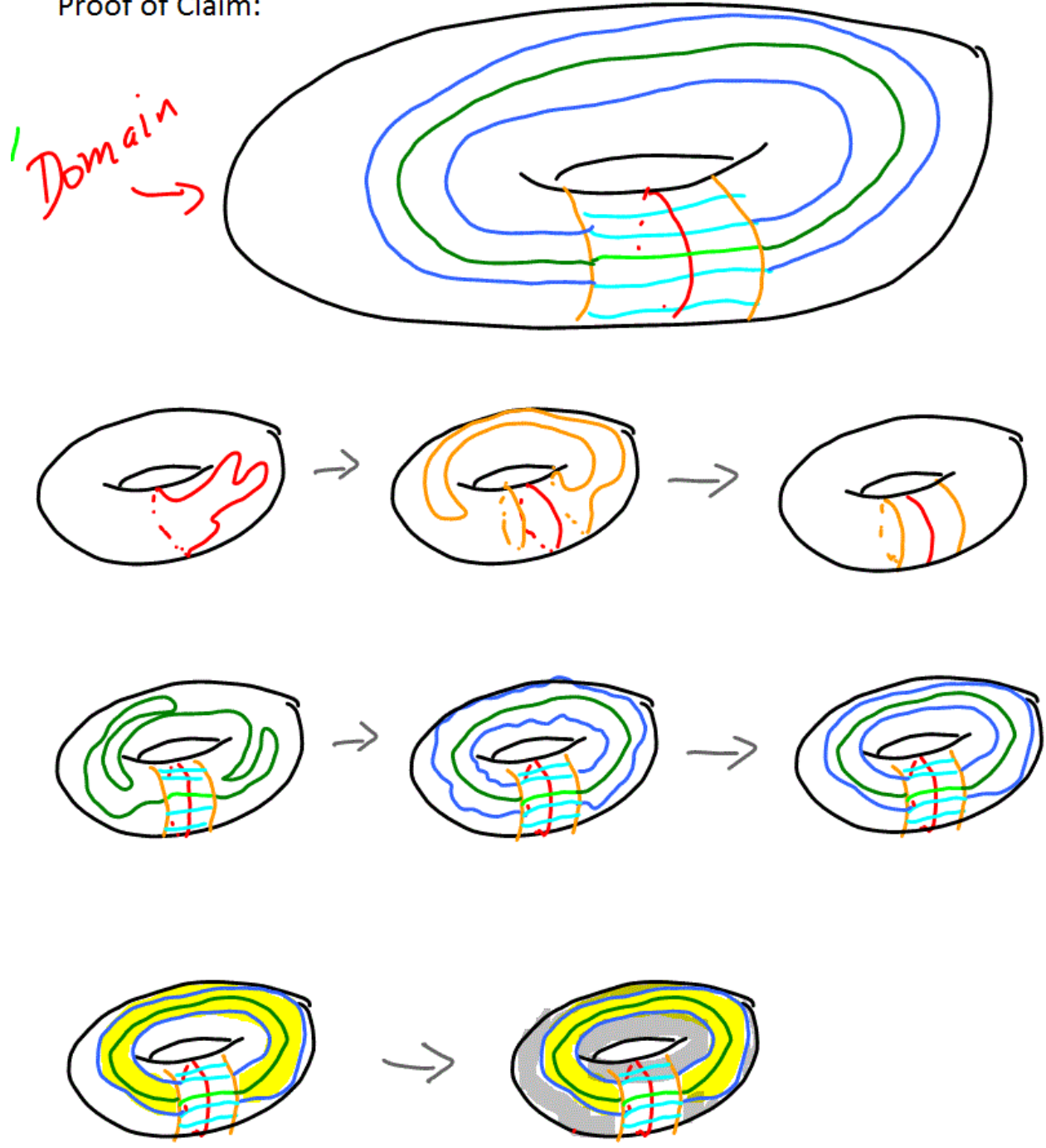
Thm 4: $\text{Aut}(T^2)/\text{ambient isotopies}$ is isomorphic to $\text{GL}(2, \mathbf{Z})$. Thus two homeomorphisms are ambient isotopic iff they have the same matrix iff they are homotopic as maps.

Pf: $*$ is surjective since $\text{GL}(2, \mathbf{Z})$ is generated by

$$h_{L^*} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad h_{M^*} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{inversion} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Claim: The kernel of $*$ is the subgroup of homeomorphisms of T^2 which are isotopic to the identity.

Proof of Claim:



2E: Solid tori.

Let $h: S^1 \times D^2 \rightarrow V$ be a homeomorphism. Then h is called a *framing* of the solid torus V .

Ex 1: If J is a s.c.c. in ∂V which is essential in ∂V , then TFAE:

- a) J is homologically trivial in V
- b) J is homotopically trivial in V
- c) J bounds a disk
- d) for some framing $h: S^1 \times D^2 \rightarrow V$, $J = h(1 \times \partial D^2)$

ambient isotopic

$$J = aL + bM = M$$

$$\begin{aligned} &\Rightarrow a = 0 \\ &\text{gcd}(a, b) = 1 \\ &\Rightarrow b = \pm 1 \end{aligned}$$

Defn: **THE** meridian of V is a s.c.c satisfying the above conditions. A *longitude* of V is ANY s.c.c of the form $h(\partial D^2 \times 1)$ for SOME framing h .

Ex 3: If K is a s.c.c. in ∂V , then TFAE:

- a.) K is a longitude of V ,
- b.) K represents a generator of $H_1(V) = \pi_1(V) = \mathbf{Z}$
- c.) K intersects some meridian of V (transversely) in a single pt.

Meridian of V : Unique up to ambient isotopy: $\langle 0, \pm 1 \rangle$.

Longitude of V : Unique up to homeomorphism. Infinitely many ambient isotopy classes of longitudes: $\langle \pm 1, b \rangle$.

$$\pm L + bM$$

Ex 5: A homeomorphism $f: \partial V \rightarrow \partial V$ extends to a self-homeo of V if and only if $f(M) = M'$ for some meridians M, M' in V .

Corollary: Suppose $\partial N = T^2$ and $h: \partial V \rightarrow \partial N$ is a homeomorphism, then up to homeomorphism, $N \cup_h V$ is determined by $h(M)$.

Pf: h is determined by $h(M)$ and $h(L)$, but by ex 5, we only need to know $h(M)$.

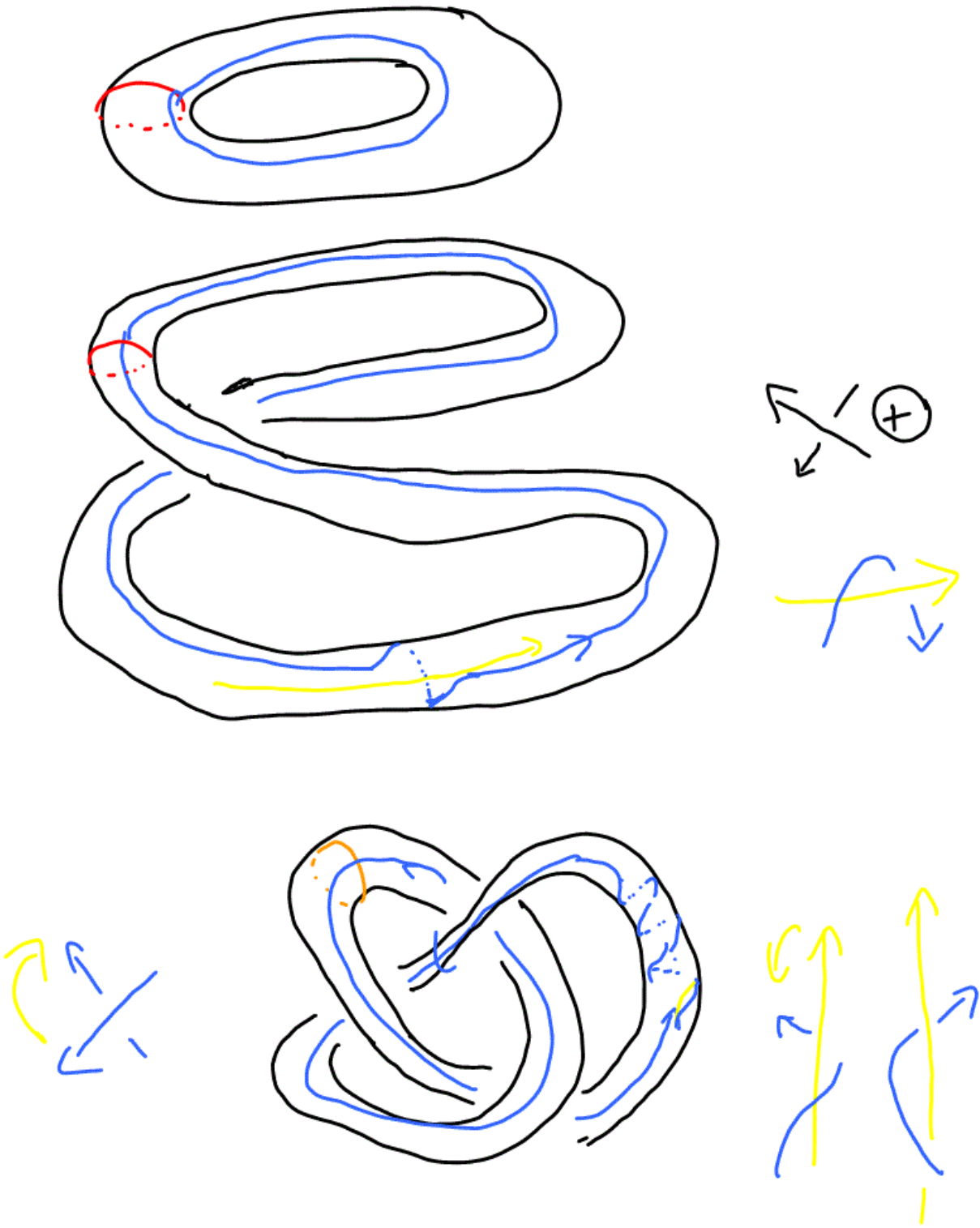
$$\begin{array}{l}
 f: \partial V \rightarrow \partial V \\
 M \rightarrow M' \\
 \hline
 \text{Id + Ext} \\
 \Rightarrow f: N \cup V \rightarrow N \cup V' \\
 \hline
 f: N \cup_h V \xrightarrow{\quad} N \cup_{h'} V' \\
 h(M) = aL + bM = h'(M') \\
 f|_N = \text{id} \Rightarrow f|_{\partial N} = \text{id} \\
 M \xrightarrow{h} aL + bM \xrightarrow{f} aL + bM \xrightarrow{(h')^{-1}} M' \\
 \hline
 \text{Ex 6: } H_1(S^3 - V) = \begin{cases} \mathbb{Z}, & i = 0, 1 \\ 0, & i \geq 2 \end{cases}
 \end{array}$$

Moreover, a generator of $H_1(S^3 - V)$ is a meridian of V

Ex 7: Up to ambient isotopy of V , there is a unique longitude which is homologically trivial in $S^3 - V$.

$$L + bM \sim cM \text{ in } S^3 - V$$

Defn: The preferred framing of V is the homeomorphism $h: S^1 \times D^2 \rightarrow V$ such that $h(S^1 \times 1)$ is homologically trivial.



Chapter 5: Seifert surfaces

Classification of 2-manifolds: Every closed orientable 2-manifold is homeomorphic to S^2 or to a connected sum of tori.

manifold	S^2	T^2	$T^2\#T^2$...	$T^2\# \dots\#T^2$
genus	0	1	2	...	g
χ	2	0	-2	...	$2-2g$

$$\chi(M) = \sum(-1)^i (\text{rank of } H_i(M)) = \sum(-1)^i (\# \text{ of } i\text{-simplices})$$

Defn: If M is an orientable manifold, $g(M) = g(\widehat{M})$ where \widehat{M} is its associated closed surface.

Ex 5A2: $g(M) = 1 - [\chi(M) + b]/2$ where $b = \#$ of ∂ components.

From **Livingston's Knot Theory**:

Thm 1, 2: If 2 connected orientable surfaces, S, S' intersect in a arc contained in their boundary, then

$$\chi(S \cup S') = \chi(S) + \chi(S') - 1$$

$$g(S \cup S') = g(S) + g(S')$$

Thm 3: Let S = a connected orientable surface formed by attaching bands to a collection of disks. Then

$$\chi(S) = \# \text{disks} - \# \text{bands}$$

$$g(S) = [2 - \# \text{disks} + \# \text{bands} - \# \text{boundary components}]/2$$

Thm 5: Every connected surface with boundary is homeomorphic to a surface constructed by attaching bands to a disk.

Thm 6: Two disks with bands attached are homeomorphic if and only if the following conditions are met:

- (1) they have the same number of bands,
- (2) they have the same number of boundary components,
- (3) both are orientable or both are nonorientable.

Suppose $X \subset Y$. X is *bicollared* in Y if there exists an embedding $b: X \times [-1, 1] \rightarrow Y$ such that $b(x, 0) = x$ for all x in X .

A *tubular neighborhood* of a 1-manifold X in a 3-manifold M is an embedding $n: X \times B^2 \rightarrow M$ such that $n(x, 0) = x$ for all x in X .

A *Seifert surface* for a link $K \subset S^3$ is a connected, bicollared, compact manifold $M^2 \subset S^3$ with $\partial M = K$.

Thm 5A4: Every knot has a Seifert surface.

Defn: If K is a knot in S^3 , then

$$g(K) = \min\{g(M) \mid M \text{ is a Seifert surface of } K\}$$

Thm 5A14: $g(K_1 \# K_2) = g(K_1) + g(K_2)$

Cor 5A17: If $K_1 \# K_2 = \text{unknot}$, then $K_1 = \text{unknot} = K_2$.

Seifert Van Kampen Thm:

Suppose $X = A \cup B$ and $Y = A \cap B$, where A, B, Y are open, nonempty and path-connected. Suppose

$$\pi_1(A) = \{ a_1, \dots, a_h \mid r_1, \dots, r_m \}$$

$$\pi_1(B) = \{ b_1, \dots, b_k \mid s_1, \dots, s_n \}$$

$$\pi_1(Y) = \{ y_1, \dots, y_p \mid t_1, \dots, t_q \}$$

Then,

$$\pi_1(X) = \{ a_1, \dots, a_h, b_1, \dots, b_k \mid r_1, \dots, r_m, s_1, \dots, s_n, \\ i(y_1) = j(y_1), \dots, i(y_p) = j(y_p) \}$$

where $i: A \rightarrow X, j: B \rightarrow X$ are inclusion maps.

$-L = L^r$ = the *reverse* of $L = L$ with the orientation of all its components reversed. L is *invertible* or *reversible* if $L = -L$.

L^* = the mirror image of L with orientation induced by the orientation reversing homeomorphism of S^3 .

L is *achiral* if $L = L^*$. L is *chiral* if $L \neq L^*$.

L is (+) *amphichiral* if $L = L^*$. L is (-) *amphichiral* if $L = -L^*$.

The inverse of $L = -L^*$

Thm: $\pi_1(K \# K') = \pi_1(-K^* \# \bullet K')$

$$3_1 \# 3_1, \quad 3_1 \# 3_1^*$$

Thm: If K, K' are prime knots, then $\pi_1(K) = \pi_1(K')$ implies $K' = K$ or K^*

For links, $S^3 - L = S^3 - L'$ does NOT imply $L = L'$, BUT see Kawauchi's A survey of knot theory.