Alexander polynomial review:

Note Title 3/22/2010

Thm: If K, K' are prime knots, then $\pi_1(K) = \pi_1(K')$ implies K' = K or K*
Recall: This is only true for knots, not links.

However, it is difficult to determine when two finitely presented groups are the same, thus instead look at invariants which can be obtained from $\pi_1(K)$.

Example: Abelianize $\pi_1(K)$ to obtain $\pi_1(K)/[\pi_1(K), \pi_1(K)] = H_1(K)$.
But $H_1(K) = \mathbf{Z}^c$, where c = number of components of K.
Hence all information except the number of components is lost.

Better example: Look at commutator subgroup of $\pi_1(K)$:

Let $C = [\pi_1(K)\ \pi_1(K)] = \{aba^{-1}b^{-1} \mid a, b \text{ in } \pi_1(K)\}$

Abelian groups are easier to work with: C/[C, C]

Better yet: Look at a Λ–module structure on C/[C, C] where

Λ is the set of Laurent polynomials = $\mathbf{Z}[t, t^{-1}]$.

I.e., the scalars are polynomials in t, t^{-1} with integer coefficients and the module elements are equivalence classes of loops in $\pi_1(K)$.

$\tilde{X}$ = universal abelian cover of X means

$$p_*: \pi_1(\tilde{X}) \to [\pi_1(X), \pi_1(X)] = C \text{ is an isomorphism.}$$

Hence $p_*: H_1(\tilde{X}) = \pi_1(\tilde{X})/[\pi_1(\tilde{X}), \pi_1(\tilde{X})] \to C/[C, C]$ is a group isomorphism.

Since universal abelian cover is regular (ie $p_*(\pi_1(\tilde{X}))$ is normal),

$$\mathrm{Aut}(\tilde{X}, p) \cong \pi_1(X)/p_*(\pi_1(\tilde{X})) \cong \pi_1(X)/[\pi_1(X), \pi_1(X)] \cong Z^c$$

where c = number of components of link.

Thus for a knot (but not a link),

universal abelian cover = infinite cyclic cover.

Can calculate infinite cyclic cover using Seifert surface or surgery description.

Since regular covering, $\tilde{x}_0, \tilde{x}_1 \in p^{-1}(x)$ implies

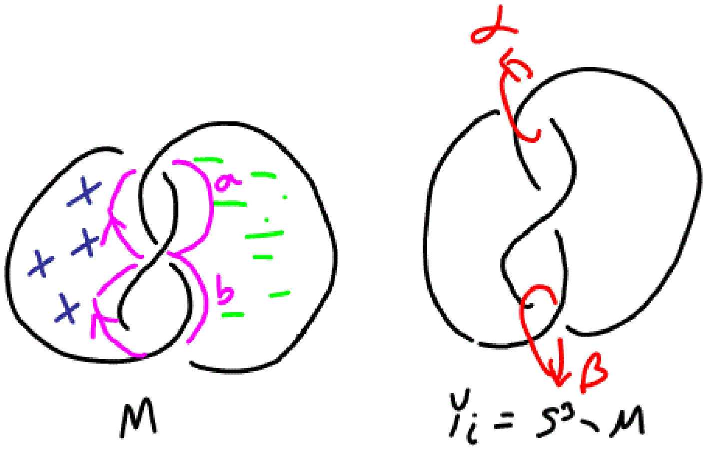
$$p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = g\, p_*(\pi_1(\tilde{X}, \tilde{x}_0))\, g^{-1} \text{ for some } g \in \pi_1(X)$$

I.e., there is a bijection between covering spaces of X and conjugacy classes of subgroups of $\pi_1(X)$

$$\begin{tikzcd}
{[\alpha]} \arrow[d] & H_1(\tilde{X}) \arrow[r, "\cong", "p_*"'] \arrow[d, "\tau_*"] & C/[C,C] \arrow[d, "t"] & {[p_*[\alpha]] = [c]} \arrow[d] \\
{[t\alpha]} & H_1(\tilde{X}) \arrow[r, "\cong", "p_*"'] & C/[C,C] & x c x^{-1}
\end{tikzcd}$$

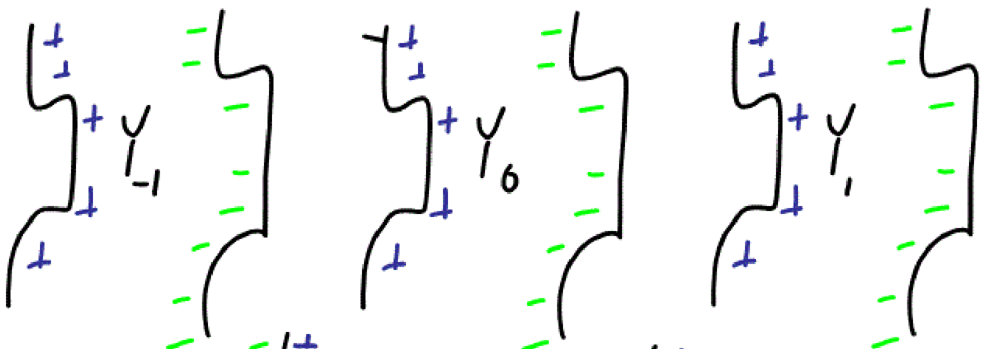
where $x \in [1] \in H_1(X)$

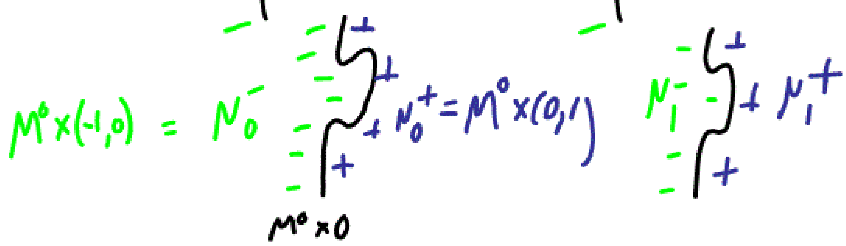
Seifert



$$a^- = \beta - \alpha$$
$$b^- = -\beta$$

$$a^+ = -\alpha$$
$$b^+ = \alpha - \beta$$



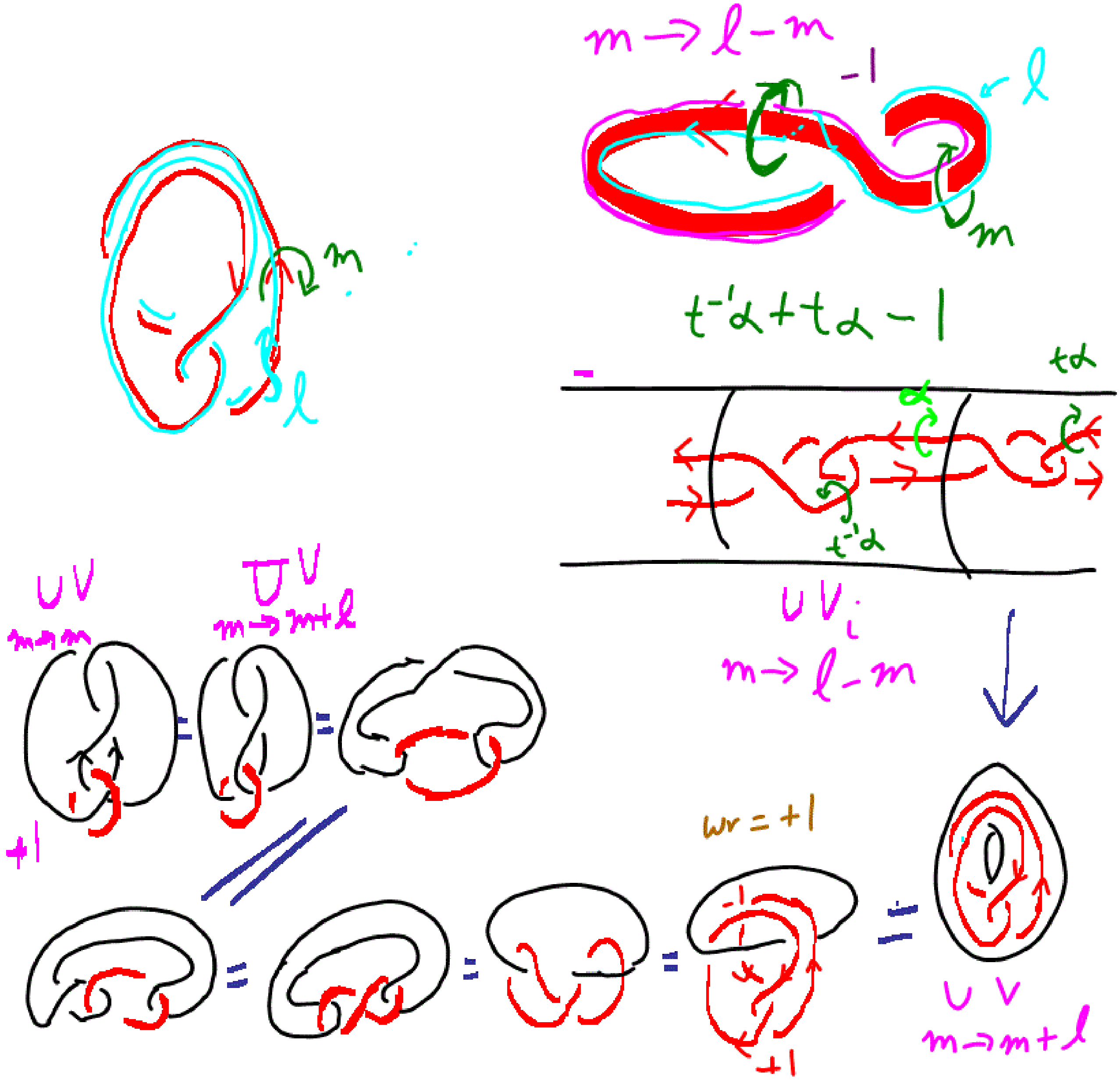


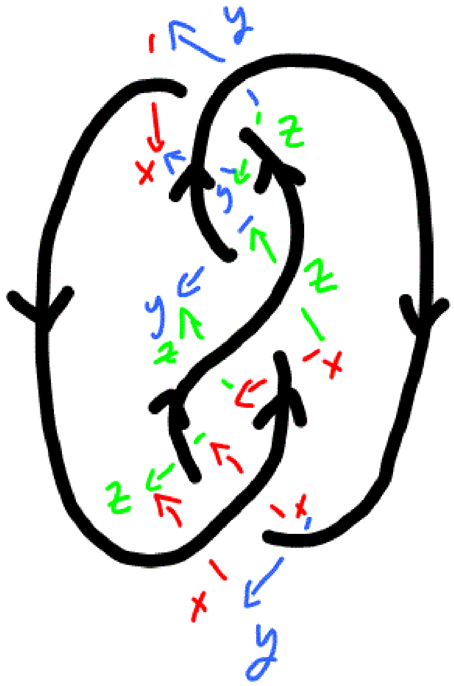
$$H_1(\tilde{X}) = \left(\alpha, \beta \;\middle|\; \begin{matrix} \beta - \alpha = -t\alpha \\ -\beta = t(\alpha - \beta) \end{matrix}\right)$$

$$-\beta = (t-1)\alpha \Rightarrow (t-1)\alpha = t\alpha(1+t-1)$$
$$0 = \alpha(t^2 - t + 1)$$

$$H_1(\tilde{X}) = \Lambda / (t^2 - t + 1)$$

Surgery description: $\Delta / (t^{-1} - 1 + t)$





$$\pi_1(S^3 - 3_1) = \left\langle x, y, z \;\middle|\; \begin{array}{l} z = x^{-1} y \; x \\ y = z^{-1} x \; z \end{array} \right\rangle$$

$$= \langle x, y \mid y = z^{-1} x \, z \rangle$$

where $z = x^{-1} y \; x$