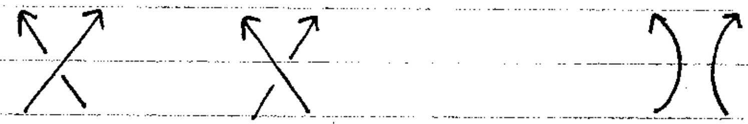
Thursday, March 24, 2010

Alexander Polynomial

The Alexander polynomial $\Delta(t)$ for a knot satisfies the skein relation

① $\Delta(L_+) - \Delta(L_-) = (t^{1/2} - t^{-1/2})\Delta L_0$



positive crossing — negative crossing — smoothed crossing

② $\Delta(O_1) = 1$

Conway Polynomial

Similar to Alexander polynomial, satisfying:

① $\nabla(L_+) - \nabla(L_-) = z\nabla(L_0)$

② $\nabla(O_1) = 1$

Note: One may get from the Conway polynomial to the Alexander polynomial by:

$$z \longrightarrow t^{1/2} - t^{-1/2}$$

Computing Alexander/Conway Polynomial for Split Link

$\therefore$ $\nabla(\text{split link}) = 0$

$\Delta(\text{split link}) = 0$

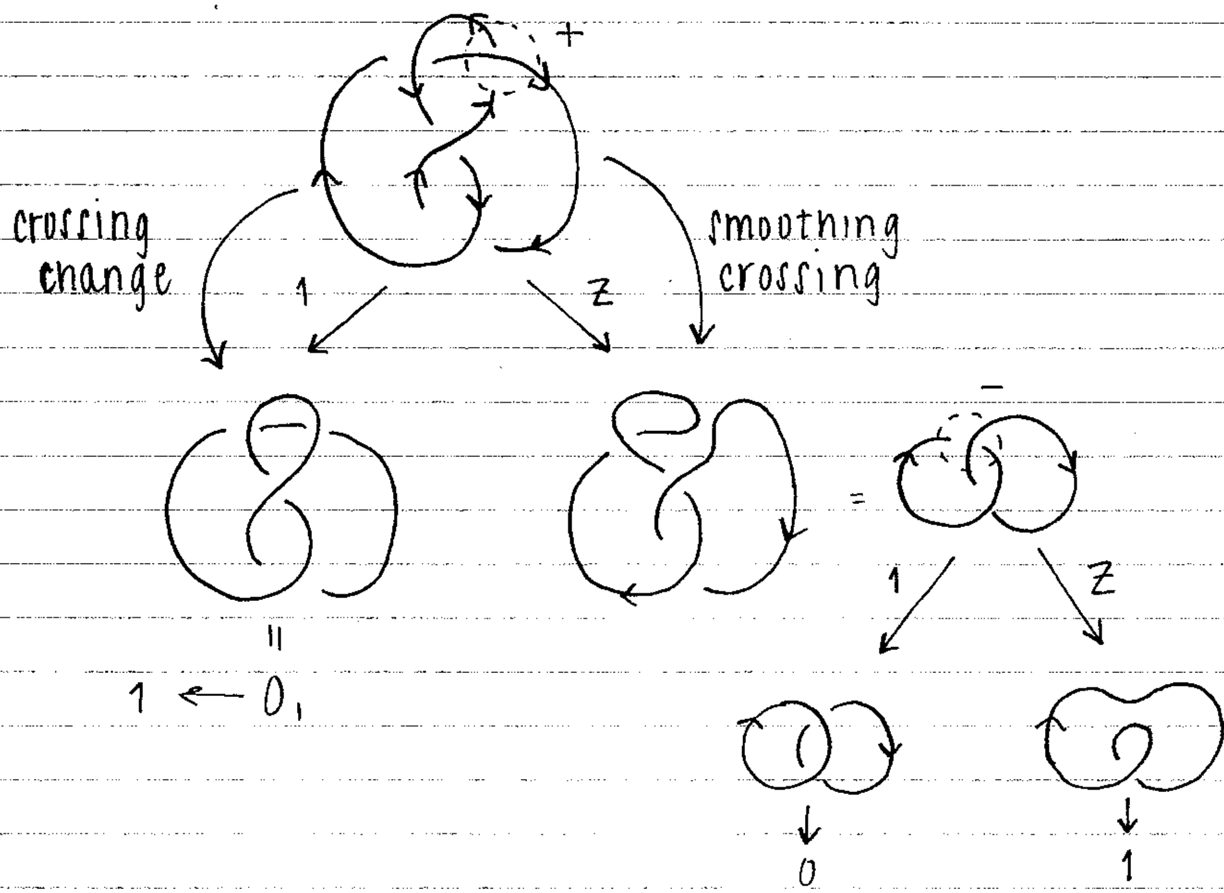
Resolving Tree

By ① of Conway polynomial, we have

$$\nabla(L_+) = \nabla(L_-) + z\nabla(L_0)$$

$$\nabla(L_-) = \nabla(L_+) - z\nabla(L_0)$$

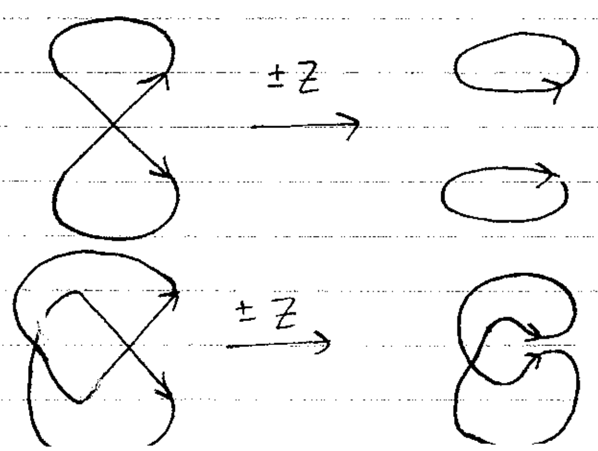
Ex: Figure Eight Knot



$$\therefore \nabla(4_1) = 1 + z\,(1 \cdot 0 + (-z)(1))$$
$$= 1 - z^2$$

$$\Delta(4_1) = 1 - (t^{1/2} - t^{-1/2})^2$$
$$= 1 - t + 2 - t^{-1}$$
$$= -t + 3 - t^{-1}$$

Ex:

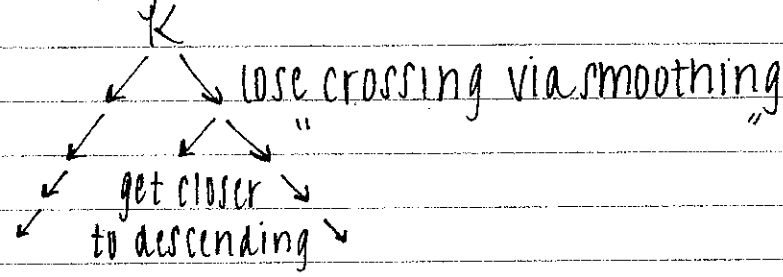


Note: We change the # of components each time we smooth a crossing.

Observe: $\Delta_K \in \begin{cases} \Lambda(t, t^{-1}) & \text{if } K \text{ is a knot} \\ \Lambda(t^{1/2}, t^{-1/2}) & \text{if } K \text{ is a link} \end{cases}$

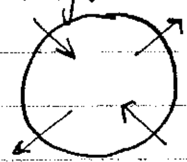
Recall: One many always change crossings of knot diagrams to obtain the unknot by making the diagram descending / ascending.

$\Rightarrow \forall L, \exists$ resolving tree

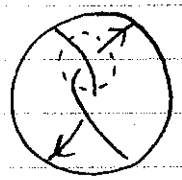
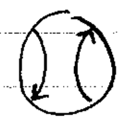
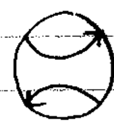


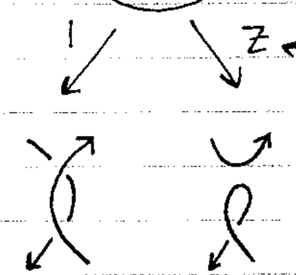
Skein Relations & Tangles

Suppose we have a tangle of the form

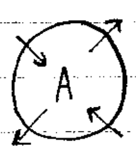
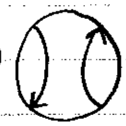
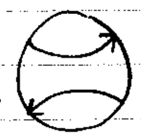


i.e. assuming oriented parity

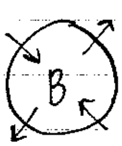
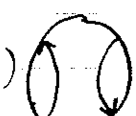
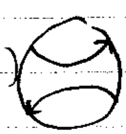
Ex:  = 1  + z 

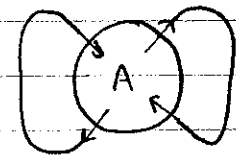
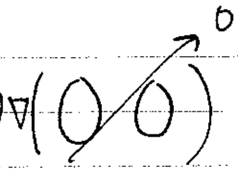
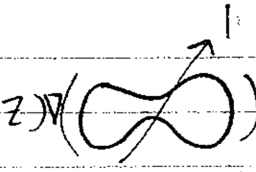


positive z since the crossing is positive

In general:  = $a_\infty(z)$  + $a_0(z)$ 

polynomials

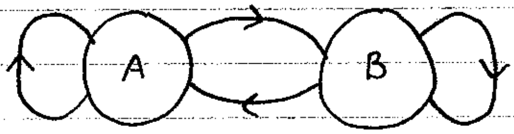
Similarly,  = $b_\infty(z)$  + $b_0(z)$ 

$D(A) =$  $= a_\infty(z)\nabla($  $) + a_0(z)\nabla($  $)$

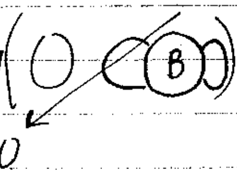
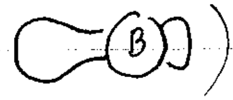
← unknot

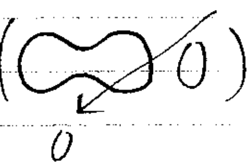
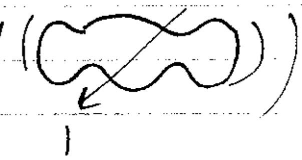
$= a_0(z)$

Let's take a look @ composite knots:



$D(A) \# D(B)$

$\Rightarrow a_\infty(z)\nabla($  $) + a_0(z)\nabla($  $)$

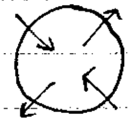
$= a_0(z)\big(b_\infty(z)\nabla($  $) + b_0(z)\nabla($  $)\big)$

$= a_0(z)\, b_0(z)\,(1)$

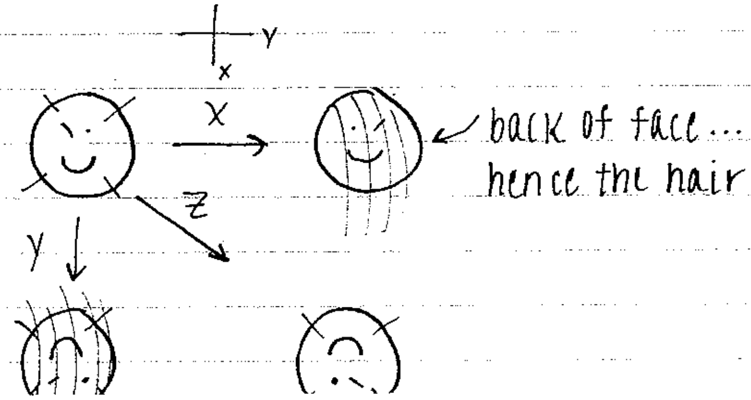
$= \nabla(a_0(z))\,\nabla(b_0(z))$

__In general:__ $\nabla(K_1 \# K_2) = \nabla(K_1)\cdot\nabla(K_2)$

$\Delta(K_1 \# K_2) = \Delta(K_1)\cdot\Delta(K_2)$

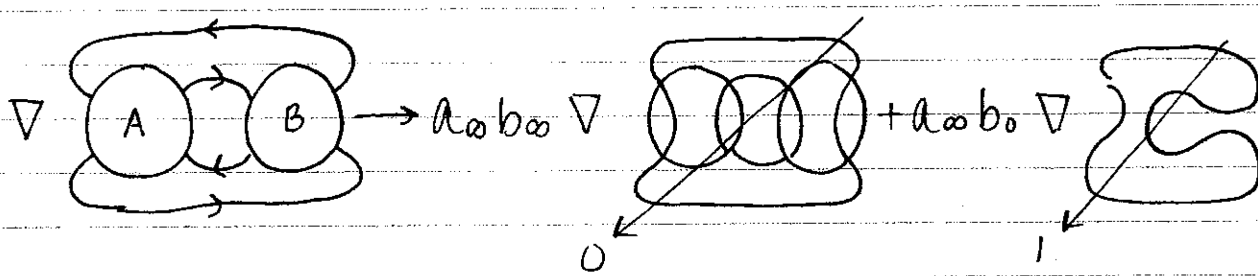
__Note:__ This result only used existence of skein relation which resolves oriented 2-string tangles of type  to)(or ⌣⌢

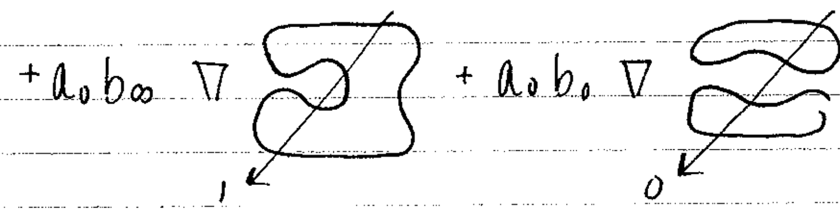
__Mutations:__



Note: The 0, ±1, & ∞ - tangles are invariant under mutations

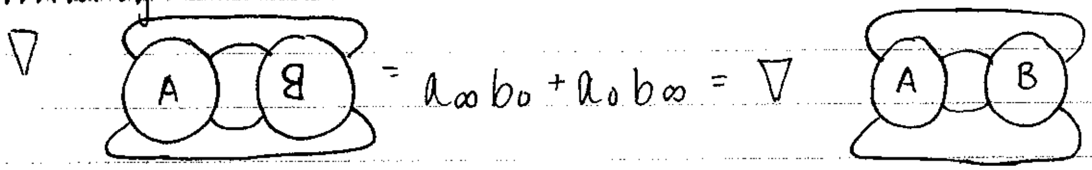
Ex: N(A+B)





$= a_\infty b_\infty + a_0 b_\infty$

Similarly:



(Similar results hold for other orientations/parities of B)

Result: Conway, Alexander, & other similarly defined polynomial invariants defined using skein relations do not detect mutations.

Homflypt Polynomial

$$a^{-1} P(L_+) - a P(L_-) = z P(L_0)$$
$$P(O_1) = 1$$

$P \longrightarrow \Delta$ by $\begin{cases} a \longrightarrow 1 \\ z \longrightarrow t^{1/2} - t^{-1/2} \end{cases}$

$P \longrightarrow$ Jones Polynomial by $\begin{cases} a \longrightarrow t \\ z \longrightarrow t^{1/2} - t^{-1/2} \end{cases}$

Facts: ① $P(K_1 \# K_2) = P(K_1) \cdot P(K_2)$
② cannot distinguish mutants

Ex: Let $a = il \Rightarrow a^{-1} = i^{-1} l^{-1} = -il^{-1}$

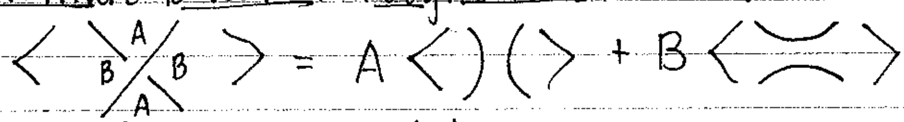
$$-il^{-1}P(L_+) - il\,P(L_-) = z\,P(L_0)$$

$$l^{-1}P(L_+) + l\,P(L_-) = iz\,P(L_0) = m\,P(L_0)$$

Note: Alexander polynomial does not detect chirality
Homflypt polynomial often detects chirality:

$$P_{K^*}(l, m) = P_K(l^{-1}, m)$$

Kauffman Bracket Polynomial (for unoriented links)



Check R2, let $B = A^{-1}$

$$\langle \bigcirc\ K \rangle = d \langle K \rangle \quad \text{where } d = -A^2 - A^{-2}$$

↑
unknotted components

Note: Invariant under R2 & R3, not R1
"invariant of framed links"