Defn: Let \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m \).

\[
\frac{\partial f_i}{\partial x_j}(a) = \lim_{h \to 0} \frac{f_i(a_1, \ldots, a_{j-1}, a_j + h, a_{j+1}, \ldots, a_n) - f_i(a_1, \ldots, a_{j-1}, a_j, a_{j+1}, \ldots, a_n)}{h}
\]

\[= \lim_{h \to 0} \frac{f_i(a + he_j) - f_i(a)}{h} \]

**Ex:** \( f : \mathbb{R}^2 \rightarrow \mathbb{R}, f(x, y) = \begin{cases} 0 & (x, y) = (0, 0) \\ \frac{xy}{x^2 + y^2} & \text{otherwise} \end{cases} \)

\[
\frac{\partial f}{\partial x}(0, 0) = \lim_{h \to 0} \frac{f(0 + h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = 0
\]

\[
\frac{\partial f}{\partial y}(0, 0) = \lim_{h \to 0} \frac{f(0, 0 + h) - f(0, 0)}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = 0
\]

BUT \( f \) is not continuous at \((0, 0)\)!!!!!!!!!!

Defn: Suppose \( f : \mathbb{R}^1 \rightarrow \mathbb{R}^1 \) is differentiable at \( a \). Then

\[
\left\{ \begin{array}{l}
\lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = f'(a) \\
\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a) \\
\quad \quad \lim_{x \to a} \frac{f(x) - f(a) - f'(a)(x - a)}{x - a} = 0 \\
\quad \quad \lim_{x \to a} \frac{f(x) - [f(a) + f'(a)(x - a)]}{(x - a)} = 0
\end{array} \right.
\]

\( y = f'(a)[x - a] + f(a) \) is the linear approximation of \( f \) near \( a \).

\[ y = f(a) + f'(a)(x-a) \] \( \approx \) \( \text{tangent line} \)

\( \approx f(x) \) near \( a \)
Defn: The gradient of \( f \) is denoted by

\[
\nabla f(a) = \left( \frac{\partial f_1}{\partial x_1}(a), ..., \frac{\partial f_1}{\partial x_n}(a) \right)
\]

Defn: The Jacobian matrix of \( f \) at \( a \) is

\[
Df(a) = \left( \frac{\partial f_i}{\partial x_j}(a) \right)_{m \times n} = \begin{pmatrix}
\frac{\partial f_1}{\partial x_1}(a) & ... & \frac{\partial f_1}{\partial x_n}(a) \\
\vdots & \ddots & \vdots \\
\frac{\partial f_m}{\partial x_1}(a) & ... & \frac{\partial f_m}{\partial x_n}(a)
\end{pmatrix}
\]

Defn: Suppose \( f : \mathbb{R}^1 \to \mathbb{R}^1 \) is differentiable at \( a \). Then

\[
limit_{h \to 0} \frac{f(a + h) - f(a) - f'(a)h}{h} = 0
\]

\[
limit_{x \to a} \frac{f(x) - f(a) - f'(a)(x - a)}{x - a} = 0
\]

Defn: Suppose \( A \subset \mathbb{R}^n, f : A \to \mathbb{R}^m \).

\( f \) is said to be **differentiable at a point** \( a \) if there exists an open ball \( V \) such that \( a \in V \subset A \) and a linear function \( T \) such that

\[
limit_{h \to 0} \frac{\|f(a + h) - f(a) - T(h)\|}{\|h\|} = 0
\]

\[
limit_{x \to a} \frac{\|f(x) - f(a) - T(x - a)\|}{\|x - a\|} = 0
\]

Near \( \bar{a} \)

\( f(x) \approx f(\bar{a}) + T^3(x - \bar{a}) \) is approximate \( f \) near \( \bar{a} \)
\[ f(x) \approx f(\bar{x}) + Df(\bar{x})(x - \bar{x}) \]

\( f : \mathbb{R}^2 \rightarrow \mathbb{R} \)

tangent (hyper) plane
Thm: \( f \) is differentiable at \( a \) implies \( f \) is continuous at \( a \).

Thm: Let \( f : \mathbb{R}^n \to \mathbb{R}^m \), \( f = (f_1, \ldots, f_m) \). \( f \) is differentiable at \( a \) iff \( f_i : \mathbb{R}^n \to \mathbb{R} \) is differentiable at \( a \) for all \( i = 1, \ldots, m \).

If \( Df(a) = \) the Jacobian evaluated at \( a \), then \( f \) is differentiable at \( a \) if the Jacobian exists. \( f \) is differentiable if the Jacobian exists and is continuous in a neighborhood of \( a \) for all \( i, j \).

Ex: Is \( f(x, y) = x^2y \) differentiable at \((3, 1)\)?

\[
Df(x, y) = \begin{pmatrix} 2xy & x^2 \end{pmatrix},
\]

\[
\frac{\partial f}{\partial x}(x, y) = 2xy \quad \text{is cont.}
\]

\[
\frac{\partial f}{\partial y}(x, y) = x^2 \quad \text{is cont.}
\]

Find the equation of the tangent plane to \( f(x, y) = x^2y \) at \((3, 1)\).

Estimate \( f(3.1, .9) \).
Tangent plane \( f(x, y) = x^2 y \) at \((3, 1)\)

\[ z = f(\hat{a}) + Df(\hat{a})(\vec{x} - \hat{a}) \]

\[ Df(x, y) = (2xy, x^2) \]

\[ Df(3, 1) = (6, 9) \]

\[ z = f(3, 1) + (6, 9) \begin{bmatrix} (x - 3) \\ (y - 1) \end{bmatrix} \]

\[ = 9 + 6(x - 3) + 9(y - 1) \]

\[ = 9 + 6x - 18 + 9y - 9 \]

\[ z = 6x + 9y - 18 \]
\[ f(3.1, 0.9) \approx \]
\[ 6(3.1) + 9(0.9) - 18 \]
\[ = 18.6 + 8.1 - 18 \]
\[ = 8.7 \]

\[ f(x, y) = x^2 y \]
\[ f(3.1^9, 0.9) = (3.1)^2(0.9) = 8.649 \]
2.4

Thm: If \( f, g : \mathbb{R}^n \to \mathbb{R}^m \) is differentiable at \( a \), then \( f + g \) is differentiable at \( a \) and \( D(f + g) = Df + Dg \).

Thm: Let \( c \in \mathbb{R} \). If \( f : \mathbb{R}^n \to \mathbb{R}^m \) is differentiable at \( a \), then \( cf \) is differentiable at \( a \) and \( D(cf)(a) = cDf(a) \).

Thm: If \( g : \mathbb{R}^n \to \mathbb{R}^m \) is differentiable at \( a \) and if \( f : \mathbb{R}^m \to \mathbb{R}^k \) is differentiable at \( g(a) \), then \( f \circ g \) is differentiable at \( a \) and \( D(f \circ g)(a) = Df(g(a))Dg(a) \).

2.5

Note for the product and quotient rule, \( f, g \) are real-valued functions; NOT vector valued.

Thm: If \( f, g : \mathbb{R}^n \to \mathbb{R} \) is differentiable at \( a \), then \( fg \) is differentiable at \( a \) and \( D(fg)(a) = g(a)Df(a) + f(a)Dg(a) \).

Thm: If \( g(a) \neq 0 \) and \( f, g : \mathbb{R}^n \to \mathbb{R} \) is differentiable at \( a \), then \( f/g \) is differentiable at \( a \) and \( D(f/g)(a) = \frac{g(a)Df(a) - f(a)Dg(a)}{g(a)^2} \).
\[ f(x, y) = \left( x^2, y \right) \]
\[ g(x, y) = \left( y \ln x, 5 \right) \]
\[ f(x, y) + g(x, y) = \left( x^2 + y \ln x, y + 5 \right) \]

\[ Df(x, y) = \begin{pmatrix} 2x & 0 \\ 0 & 1 \end{pmatrix} \]

\[ Dg(x, y) = \begin{pmatrix} y/x & \ln x \\ 0 & 0 \end{pmatrix} \]

\[ D(f+g) = \begin{pmatrix} 2x + y/x & 0 + \ln x \\ 0 + 0 & 1 + 0 \end{pmatrix} \]

\[ D\left(5f\right) = 5Df = 5\begin{pmatrix} 2x & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 10x & 0 \\ 0 & 5 \end{pmatrix} \]

\[ 5f(x, y) = (5x^2, 5y) \]
\[ f(x, y) = (\sin(x+y), x^2y^3) \]
\[ g(x, y) = \left( e^{xy}, \ln(x^2+y) \right) \]
\[ f \circ g(x, y) = \left( \sin(x+y) e^{xy}, x^2y^3 \ln(x^2+y) \right) \]

\[ D(f \circ g)(x, y) = \begin{pmatrix}
\cos(x+y)e^{xy} + ye^{xy}\sin(x+y)
\end{pmatrix} \]
$$Df(x,y) = \begin{pmatrix} \cos(x+y) \ e^{xy} + ye^{xy} \sin(x+y) \\ 2xy^3 \ln(x+y) + x^2y^3 - \frac{2x}{x^2+y} \\ 3x^2y^2 \ln(x+y) + x^2y^3 - \frac{1}{x^2+y} \end{pmatrix}$$
\((f_9), = \sin(x+y) e^{xy}\)

\(f_1(x,y) = \sin(x+y)\)

\(g_1(x,y) = e^{xy}\)

\[D((f_9)_1) = g(x,y) Df(x,y) + f(x,y) Dg(x,y)\]

\[= e^{xy} \left( \cos(x+y), \cos(x+y) \right) + \sin(x+y) \left( ye^{xy}, xe^{xy} \right)\]