1.1 Vectors:
Let \( \mathbf{v} = \left( \frac{1}{2} \right) \).

If \( \mathbf{v} \) = velocity in m/sec of an object, then the object is moving east at a rate of 1 m/sec and north at a rate of 2 m/sec.

Speed of the object =
\[
\|
\mathbf{v}
\| = \|\mathbf{v}\| = \sqrt{1^2 + 2^2}
\]

A vector can be described by its Euclidean coordinates OR by its length and direction.

Let \( \mathbf{w} = \left( \frac{3}{1} \right) \). Then \( \mathbf{v} + \mathbf{w} = \left( \frac{1}{2} \right) + \left( \frac{3}{1} \right) = \left( \frac{1+3}{2+1} \right) = \left( \frac{4}{1} \right) \).

\[ \mathbf{v} + \mathbf{w} = \mathbf{0} \text{ at } (1, 2) \text{ and } (2, 1) \]

\[ \mathbf{v} - \mathbf{w} = \left( \frac{1}{2} \right) - \left( \frac{3}{1} \right) = \left( \frac{1}{2} \right) + \left( \frac{-3}{1} \right) = \left( \frac{1-3}{2+1} \right) = \left( \frac{-2}{3} \right) \]

\[ \mathbf{v} - \mathbf{w} \text{ is the vector starting at the point } \begin{pmatrix} 1 \\ 3 \end{pmatrix} \text{ and ending at the point } \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

2.1 Let \( f : X \to Y \) where \( X \subseteq \mathbb{R}^n, Y \subseteq \mathbb{R}^m \)

Graph of \( f = \{(x, f(x)) | x \in X \} \subseteq \mathbb{R}^n \times \mathbb{R}^m \)

Domain of \( f = X \),

Ccodomain of \( f = Y \)

Range of \( f = \text{Image of } f = f(X) \)

\[ \{ y \in Y | \text{ there exists } x \in X \text{ such that } f(x) = y \} \]

\( f \) is a function if for all \( x \) in domain of \( f \), \( f(x) \) has a unique value.

I.e., for all \( x, y \) in \( Y \), if \( x = y \), then \( f(x) = f(y) \) and for all \( x \) in \( X \), \( f(x) \) is defined.

\[ f \text{ is 1:1 if } f(x) = f(y) \text{ implies } x = y. \]

\( f \) gives a one-to-one correspondence between \( X \) and \( f(X) \).

Given \( b \in Y \), \( f(x) = b \) has at most one solution

Side-note: \( f(x) = b \) has exactly one solution if \( b \in f(X) \).

Side-note: \( f(x) = b \) has no solution if \( b \notin f(X) \).

\[ f \text{ is onto if } f(X) = Y \text{ (i.e., image of } f \text{ = codomain of } f). \]

Given \( b \in Y \), \( f(x) = b \) has at least one solution.

\[ f(x) = x^2 \quad f : \mathbb{R} \to \mathbb{R} \]

\[ f(x) = -y \quad f : \mathbb{R} \to \mathbb{R} \]

Not onto

\[ f(x) = x^2 \]

\[ f(x) = -y \]

\[ f : \mathbb{R} \to \mathbb{R} \]
Ex 1: \( f: \mathbb{R}^n \rightarrow \mathbb{R}, f(x) = ||x|| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2} \)

Domain = \( \mathbb{R}^n \)  
Codomain = \( \mathbb{R} \)  
Image = \([0, \infty)\)

Is \( f \) 1:1? NO

Proof: \( f(1, 0, 0, \ldots, 0) = \sqrt{1+0+0+\ldots} = 1 \)
\( f(-1, 0, \ldots, 0) = \sqrt{1+0+0+\ldots} = 1 \)

Is \( f \) onto? NO

Proof: \( \text{Codomain} = \mathbb{R} \)  
\( \text{Image} = [0, \infty) \)  
\( \mathbb{R} \neq [0, \infty) \)

Alternate Proof:
\( f(-x) = -1 \) has no solution or -1 is not in the image of \( f \)

Ex 2: \( g(x, y) = (x^2y^4 - y^6) \)

Domain = \( \mathbb{R}^2 \)  
Codomain = \( \mathbb{R} \)

Is \( g \) 1:1? NO

Proof: \( g(4, 0) = g(-4, 0) \)

Alternate Proof: \( g(0, 1) = g(0, -1) \)

Is \( g \) onto? NO

Proof: \( g(x, y) = (0, 0, -1) \) has no solution \( (0, 0, -1) \) is not in image of \( g \)

Ex 3: \( h(x) = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 2y + 3z \\ 4x + 5y + 6z \end{pmatrix} \)

I.e., \( h(x) = (x + 2y + 3z, 4x + 5y + 6z) \).

Domain = \( \mathbb{R}^3 \)  
Codomain = \( \mathbb{R}^2 \)  
Image = \([0, \infty)\)

Is \( h \) onto? NO

Is \( h \) 1:1?

How many solutions does \( h(x) = b \) have?

I.e., how many solutions does \( \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \) have?

I.e., how many solutions does the following system of equations have:
\[
\begin{align*}
x + 2y + 3z &= b_1, \\
4x + 5y + 6z &= b_2.
\end{align*}
\]

Does \( \begin{pmatrix} 1 \\ 4 \\ 5 \\ 6 \end{pmatrix} x + \begin{pmatrix} 2 \\ 2 \\ 3 \\ 6 \end{pmatrix} y + \begin{pmatrix} 3 \\ 6 \end{pmatrix} z \) span all of \( \mathbb{R}^2 \)?

Is \( \{ \begin{pmatrix} 1 \\ 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 3 \\ 6 \end{pmatrix} \} \) linearly independent?
**Definitions:**

If the codomain of $f$ is $\mathbb{R}$ (i.e., $f : X \rightarrow \mathbb{R}$), we say that $f$ is real-valued or scalar valued.

Suppose $f : X \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ and $c$ is a constant scalar.

The **level curve at height** $c$ of $f$ is the curve in $\mathbb{R}^2$ defined by $f(x, y) = c$. That is, the level curve at height $c$ of $f = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = c\}$.

The **contour curve at height** $c$ of $f$ is the curve in $\mathbb{R}^3$ defined by the two equations, $z = f(x, y), z = c$. That is, the contour curve at height $c$ of $f = \{(x, y, z) \in \mathbb{R}^2 \mid z = f(x, y) = c\} = \{(x, y, f(x, y)) \in \mathbb{R}^3 \mid f(x, y) = c\}$.

Recall the graph of $f = \{(x, y, z) \mid z = f(x, y)\}$

= $\{(x, y, f(x, y)) \mid (x, y) \in X\} \subset \mathbb{R}^2 \times \mathbb{R}$

The **section of the graph of** $f$ by the plane $z = c$ is the set of points in $\mathbb{R}^3$ defined by the two equations, $z = f(x, y), x = c$. That is, the section by $x = c$ is $\{(x, y, z) \in \mathbb{R}^2 \mid z = f(x, y), x = c\}$

= $\{(c, y, f(c, y)) \in \mathbb{R}^3 \mid (c, y) \in X\}$.

The section by $y = c$ is $\{(x, y, z) \in \mathbb{R}^2 \mid z = f(x, y), y = c\}$

= $\{(x, c, f(x, c)) \in \mathbb{R}^3 \mid (x, c) \in X\}$.

**Graph of** $f : \mathbb{R}^3 \rightarrow \mathbb{R}$

The domain is $\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$

$0 \mid 0 \mid 0 \mid 0 \mid 0 \mid 0$
Parabola: \( y = ax^2 + b \)

\[ \frac{y - \frac{a}{4}}{2} = x \]

Hyperbola: \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \)

Ellipse: \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \)

Circle: \( x^2 + y^2 = r^2 \)

In 2D: \( Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \)
Conics in $\mathbb{R}^2$: $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

for suitable constants $A, \ldots, F$.

In $\mathbb{R}^3$, the analytic analogue of the conic section is called a **quadric surface**. Quadric surfaces are those defined by equations that are polynomials of degree two in three variables:

$$Ax^2 + Bxy + Cxz + Dy^2 + Eyz + Fz^2 + Gx + Hy + Iz + J = 0.$$ 

**Figure 2.20** The sphere of radius $a$, centered at $(x_0, y_0, z_0)$.

**Figure 2.21** The ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$
**Figure 2.22** The elliptic paraboloid

\[
\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}.
\]

\[
z = k' + \frac{y^2}{z}.
\]

**Figure 2.23** The hyperbolic paraboloid

\[
\frac{z}{c} = \frac{y^2}{b^2} - \frac{x^2}{a^2}.
\]
Figure 2.24 The elliptic cone \( \frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2} \).

Section at \( z = c \):
- Contour curve at height \( c \).

\( x = 3 \) hyperbola: \( \frac{z^2}{c^2} - \frac{y^2}{b^2} = \frac{9}{a^2} \)

Figure 2.25 The graph of the equation \( \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \) is a hyperboloid of one sheet.

Figure 2.26 The graph of the equation \( \frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \) is a hyperboloid of two sheets.

Larger \( |z| \)