6.2 Green's Theorem

Suppose $F : \mathbf{R}^2 \to \mathbf{R}^2$, $F(x,y) = (M(x,y), N(x,y))$.
$S \subset \mathbf{R}^2$ is a nice compact surface.

$\int_{\partial S} F \cdot ds = \int_{\partial S} (M, N) \cdot (dx, dy) =$

$\int_{\partial S} M dx + N dy = \int\int_S [\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}] dA = \int\int_S [\nabla \times F] \cdot \mathbf{k} dA$

7.3 Stoke's Theorem

Suppose $F : \mathbf{R}^3 \to \mathbf{R}^3$, $F(x,y,z) = (M(x,y,z), N(x,y,z), P(x,y,z))$.
$S \subset \mathbf{R}^3$ is a bounded, piecewise smooth, oriented surface.

$$\int_{\partial S} F \cdot ds = \int\int_S [\nabla \times F] \cdot dS$$
$$= \int\int_X [(\nabla \times F)(X(s,t))] \cdot N(s,t) ds dt$$

Ex: Suppose $\partial S = \emptyset$

$\int\int_S [\nabla \times F] \cdot dS = \int_{\partial S} F \cdot ds =$

Ex: Suppose $\partial S = \{(x,y,0) \mid x^2 + y^2 = 1\} = \partial \tilde{S}$

$\int\int_S [\nabla \times F] \cdot dS = \int_{\partial S} F \cdot ds = \int\int_{\tilde{S}} [\nabla \times F] \cdot d\tilde{S}$

Suppose $\tilde{S} = \{(x,y,0) \mid x^2 + y^2 \leq 1\}$

A parametrization for $\tilde{S}$

$X(s,t) = (s(cost), s(sint), 0)$

$T_s(s,t) = (cost, sint, 0)$

$T_t(s,t) = (-s(sint), s(cost), 0)$

$N(s,t) = (0, 0, s)$

Suppose $F(x,y,z) = (y^2, 2x, z^2)$

Another parametrization for $\tilde{S}$

$Y(s,t) = (s,t,0)$

$T_s(s,t) = (1,0,0)$

$T_t(s,t) = (0,1,0)$

$N(s,t) = (0,0,1)$

Suppose $F(x,y,z) = (y^2, 2, z^2)$

$\nabla \times F = (0,0,-2y)$

$\int\int_{\tilde{S}} [\nabla \times F] \cdot d\tilde{S} = \int\int_{\tilde{S}} (0,0,-2y) \cdot d\tilde{S}$

$= \int\int_Y [(\nabla \times F)(X(s,t))] \cdot N(s,t) dsdt$

$= \int_{-1}^{1} \int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} (0,0,-2t) \cdot (0,0,1) dsdt$

$= -\int_{-1}^{1} \int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} \ 2tdsdt$

$= -\int_{-1}^{1} 2ts|_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} \ dt$

$= -\int_{-1}^{1} 4t\sqrt{1-t^2} \ dt$

Let $u = 1 - t^2$, $du = -2tdt$, $t = \pm 1$ implies $u = 0$

$= -\int_0^0 -2\sqrt{u} \ du = 0$

6.2 Divergence Theorem in the Plane

Suppose $F : \mathbf{R}^2 \to \mathbf{R}^2$, $F(x, y) = (M(x, y), N(x, y))$.
$S \subset \mathbf{R}^2$ is a nice compact surface.
$\mathbf{n} \subset \mathbf{R}^2$, outward unit normal to ∂S.

$\int_{\partial S} F \cdot \mathbf{n} ds = \int\int_S [\nabla \cdot F] dA$

7.3 Gauss's Theorem

Suppose $F : \mathbf{R}^3 \to \mathbf{R}^3$, $F(x, y, z) = (M(x, y, z), N(x, y, z), P(x, y, z))$.
$D \subset \mathbf{R}^3$ is a bounded, solid nice 3-dimensional region.

$\int\int_{\partial D} F \cdot dS = \int\int\int_D [\nabla \cdot F] dV$

Calculate $\int\int_S F \cdot dS$ where

$F(x, y, z) = (xy^2, y^3, 4x^2 z), \quad S = S_1 \cup S_2 \cup S_3$ and

$S_1 = \{(x, y, 5) \mid x^2 + y^2 \leq 4\}$

$S_2 = \{(x, y, z) \mid x^2 + y^2 = 4, 0 \leq z \leq 5\}$

$S_3 = \{(x, y, 0) \mid x^2 + y^2 \leq 4\}$

Let S_a = sphere of radius a centered at the point $P = (P_1, P_2, P_3)$

Prop 3.4. Divergence of F at P=

$$(\nabla \cdot F)(P_1, P_2, P_3) = lim \tfrac{3}{4\pi a^3} \int\int_{S_a} F \cdot dS$$

$$= lim_{a \to 0^+} \frac{\int\int_{S_a} F \cdot dS}{\frac{4\pi a^3}{3}}$$

$$= lim_{a \to 0^+} \frac{\text{flux}}{\text{volume of ball of radius a}} = \text{flux density}$$

Let C_a = the circle of radius a centered at the point $P = (P_1, P_2, P_3)$ lying in the plane perpendicular to the unit vector $\mathbf{n}$ and containing the point P.

Prop 3.5. The component of the curl of F at P in the direction of $\mathbf{n}$ =

$$\mathbf{n} \cdot (\nabla \times F)(P) = lim_{a \to 0^+} \tfrac{1}{\pi a^2} \int_{C_a} F \cdot ds$$

$$= lim_{a \to 0^+} \tfrac{1}{\pi a^2} \int_X (F \cdot T) ds$$

$$= \frac{\text{circulation of } F \text{ along } C_a}{\text{area of surface bounded by } C_a}$$