Scalar Line Integrals:

Let \( \mathbf{x} : [a, b] \to \mathbb{R}^n \) be a \( C^1 \) path. \( f : \mathbb{R}^n \to \mathbb{R} \), a scalar field.

\[
\Delta s_k = \text{length of kth segment of path} = \int_{t_{k-1}}^{t_k} ||\mathbf{x}'(t)|| \, dt = ||\mathbf{x}'(t_k^*)|| (t_k - t_{k-1}) = ||\mathbf{x}'(t_k^*)|| \Delta t_k
\]

for some \( t_k^* \in [t_{k-1}, t_k] \)

\[
\int_{\mathbf{x}} f \, ds \sim \sum_{i=1}^n f(\mathbf{x}(t_i^*)) \Delta s_k = \sum_{i=1}^n f(\mathbf{x}(t_k^*)) ||\mathbf{x}'(t_k^*)|| \Delta t_k
\]

Thus \( \int_{\mathbf{x}} f \, ds = \int_a^b f(\mathbf{x}(t)) ||\mathbf{x}'(t)|| \, dt \)

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Vector Line integrals:

Let \( \mathbf{x} : [a, b] \to \mathbb{R}^n \) be a \( C^1 \) path. \( F : \mathbb{R}^n \to \mathbb{R}^n \), a vector field.

\[
\mathbf{x}'(t_k^*) \sim \frac{\Delta \mathbf{x}_k}{\Delta t_k}
\]

\[
\int_{\mathbf{x}} F \cdot ds \sim \sum_{i=1}^n F(\mathbf{x}(t_k^*)) \cdot \Delta \mathbf{x}_k = \sum_{i=1}^n F(\mathbf{x}(t_k^*)) \cdot \mathbf{x}'(t_k^*) \Delta t_k
\]

Thus \( \int_{\mathbf{x}} F \cdot ds = \int_a^b F(\mathbf{x}(t)) \cdot \mathbf{x}'(t) \, dt \)
Other Formulations of Vector Line integrals:

The tangent vector to $\mathbf{x}$ at $t$ is $T(t) = \frac{\mathbf{x}'(t)}{||\mathbf{x}'(t)||}$

\[\int_{\mathbf{x}} F \cdot ds = \int_{a}^{b} F(\mathbf{x}(t)) \cdot \mathbf{x}'(t) dt = \int_{a}^{b} F(\mathbf{x}(t)) \cdot \mathbf{x}'(t) \frac{||\mathbf{x}'(t)||}{||\mathbf{x}'(t)||} dt = \int_{0}^{b} F(\mathbf{x}(t)) \cdot T(t) ||\mathbf{x}'(t)|| dt = \int_{\mathbf{x}} F(\mathbf{x}(t)) \cdot T(t) ds\]

Note $\int_{a}^{b} F(\mathbf{x}(t)) \cdot T(t) ds$ is a scalar line integral of the scalar field $F \cdot T : \mathbb{R}^n \rightarrow \mathbb{R}$ over the path $\mathbf{x}$.

Note: $F \cdot T$ is the tangential component of $F$ along the path $\mathbf{x}$.

Another notation (differential form):

For simplicity, we will work in $\mathbb{R}^2$, but the following generalizes to any dimension.

Let $\mathbf{x}(t) = (x(t), y(t))$. Let $F(x, y) = (M(x, y), N(x, y))$

$x = x(t), y = y(t)$. Also $x'(t) = \frac{dx}{dt}, y'(t) = \frac{dy}{dt}$

\[\int_{\mathbf{x}} F \cdot ds = \int_{a}^{b} F(\mathbf{x}(t)) \cdot \mathbf{x}'(t) dt\]

\[= \int_{a}^{b} (M(x, y), N(x, y)) \cdot (x'(t), y'(t)) dt\]

\[= \int_{a}^{b} M(x, y)x'(t) dt + N(x, y)y'(t) dt\]

\[= \int_{\mathbf{x}} M(x, y)dx + N(x, y)dy\]
Definitions:

A *curve* is the image of piecewise $C^1$ path $x : [a, b] \rightarrow \mathbb{R}^n$.

A curve is *simple* if it has no self-intersections; that is, $x$ is 1:1 on the open interval $(a, b)$

A path is *closed* if $x(a) = x(b)$

A curve is *closed* if $x(a) = x(b)$

$\int_x F \cdot ds$ is called the circulation of $f$ along $x$ if $x$ is a closed path.
A parametrization of a curve \(C\) is a path whose image is \(C\). Normally we will require a parametrization of a curve to be 1:1 where possible.

A piecewise \(C^1\) path \(y : [c, d] \rightarrow \mathbb{R}^n\) is a reparametrization of a piecewise \(C^1\) path \(x : [a, b] \rightarrow \mathbb{R}^n\) if there exists a bijective \(C^1\) function \(u : [c, d] \rightarrow [a, b]\) where the inverse of \(u\) is also \(C^1\) and \(y = x \circ u\) (i.e., \(y(t) = x(u(t))\)).

Note that either
1.) \(u(a) = c\) and \(u(b) = d\). In this case, we say that \(y\) (and \(u\) are orientation-preserving OR
2.) \(u(a) = d\) and \(u(b) = c\). In this case, we say that \(y\) (and \(u\) are orientation-reversing.

Given piecewise \(C^1\) path, \(x : [a, b] \rightarrow \mathbb{R}^n\), the opposite path is \(x_{opp} : [a, b] \rightarrow \mathbb{R}^n\) \(x_{opp} = x(a + b - t)\)

That is \(x_{opp}\) is an orientation-reversing reparametrization of \(x\) where \(u[a, b] \rightarrow [a, b], u(t) = a + b - t\).

Thm: Let \(x : [a, b] \rightarrow \mathbb{R}^n\) be a piecewise \(C^1\) path and let \(y : [c, d] \rightarrow \mathbb{R}^n\) be a reparametrization of \(x\). Then

if \(f : \mathbb{R}^n \rightarrow \mathbb{R}\) is continuous, then \(\int_y f \cdot ds = \int_x f \cdot ds\)

if \(F : \mathbb{R}^n \rightarrow \mathbb{R}^n\) is continuous, then

\[
\int_y F \cdot ds = \int_x F \cdot ds \text{ if } y \text{ is orientation-preserving.}
\]

\[
\int_y F \cdot ds = -\int_x F \cdot ds \text{ if } y \text{ is orientation-reversing.}
\]