Real-valued function:

If \( f : \mathbb{R}^n \to \mathbb{R} \), then \( Df = \nabla f = (\partial f / \partial x_1, ..., \partial f / \partial x_n) \)

Ch 3: Path in \( \mathbb{R}^n \):

If \( x : I \subset \mathbb{R} \to \mathbb{R}^n \), then \( Dx = [x'(t)]^T \)

If \( x \) represents position,

\( v(t) = x'(t) \) = velocity vector and

\( a(t) = v'(t) = x''(t) \) = acceleration vector.

Speed of path = \( ||x'(t)|| \)

If \( x : [a, b] \to \mathbb{R}^n \), then

length of path = \( L(x) = \int_a^b ||x'(t)|| \, dt \)

Arclength parameter = \( s : [0, L(x)] \to [a, b] \)

\( s(t) = t\sqrt{17} - 10\sqrt{17} \)

Example: \( x : [10, 100] \to \mathbb{R}^2, x(t) = (t, 4t) \)
velocity vector = \( v(t) = x'(t) = (1, 4) \) and
acceleration vector = \( a(t) = v'(t) = x''(t) = (0, 0) \)
Speed = \( ||x'(t)|| = ||(1, 4)|| = \sqrt{1^2 + 4^2} = \sqrt{17} \)
length of path = \( L(x) = \int_{10}^{100} ||x'(t)|| \, dt = \int_{10}^{100} \sqrt{17} \, dt \)
\( = \sqrt{17} |_{10}^{100} = \sqrt{17}(100 - 10) = 90\sqrt{17} \)
Arclength parameter = \( s : [10, 100] \to [0, 90\sqrt{17}] \)

\( s(t) = \int_{10}^{t} \sqrt{17} \, du = \sqrt{17} |_{10}^{t} = \sqrt{17}(t - 10) \)
Thus \( s : [10, 100] \to [0, 90\sqrt{17}] \)

\( s(t) = t\sqrt{17} - 10\sqrt{17} \)
Change of parametrization:

Solve for $t$ to find $s^{-1}$: $t = \frac{s + 10\sqrt{17}}{\sqrt{17}}$.

Let the function $s^{-1}$ be denoted by $t$

Hence $t : [0, 90\sqrt{17}] \rightarrow [10, 100]$, $t(s) = \frac{s + 10\sqrt{17}}{\sqrt{17}}$.

$y : [0, 90\sqrt{17}] \rightarrow \mathbb{R}^2$, $y(s) = (x \circ t)(s)$

Thus the reparametrization of the path $x$ using the arclength parameter $s$ is

$y(s) = x(t(s)) = x(\frac{s + 10\sqrt{17}}{\sqrt{17}}) = (\frac{s + 10\sqrt{17}}{\sqrt{17}}, 4(\frac{s + 10\sqrt{17}}{\sqrt{17}}))$

Observe $\|y'(s)\| = 1$ and the length of the path traveled by time $s$ is $\int_0^s \|y'(u)\|du = s$

Unit tangent vector: $T(t) = \frac{x'(t)}{\|x'(t)\|}$

Note $T(t)$ is perpendicular to $T'(t)$

Outline of proof: Use $\frac{d}{dt} \|T(t)\|^2 = \frac{d}{dt} (1)$ to show $T(t) \cdot T'(t) = 0$

Let unit normal vector $= N = \frac{T'(t)}{\|T'(t)\|}$

Let unit binormal vector $B = T \times N$.

Note $B$ is a unit vector.

$T(t)$, $N(t)$, $B(t)$ is a set of mutually orthogonal unit vectors called the moving frame (or Frenet frame).

Curvature $\kappa = \|\frac{dT}{ds}\| = \|\frac{dT}{dt}\| \frac{dt}{ds}$

Torsion $\tau: \|\frac{dB}{ds}\| = \|\frac{dB}{dt}\| \frac{dt}{ds} = \tau N$

Intrinsic quantity: does NOT depend on parametrization:

Ex: $T$, $N$, $B$, $\kappa$, $\tau$, length of path.

Extrinsic quantity: depends on parametrization:

Ex: speed, velocity, acceleration