Calculus I review:

Suppose \( f : \mathbb{R} \rightarrow \mathbb{R} \).

Recall the tangent line to \( y = f(x) \) at \( x = a \) is 
\[
y = f(a) + f'(a)(x - a).
\]

Thus \( y = f(a) + f'(a)(x - a) \) is the best linear approximation of \( f \) near \( x = a \).

Ex: Find the best linear approximation for \( f(x) = 2x + 5 \).

Answer:

Note slope = \( f'(x) = 2 \).

Ex: Find the best linear approximation for \( h(x) = x^2 \) at \( x = 3 \).

\[ h'(x) = 2x. \]

Thus slope of tangent line at \( x = 3 \) is \( h'(3) = 2(3) = 6 \).

Hence \( \frac{y - 9}{x - 3} = 6 \)

Equation of tangent line: \( y = 9 + 6(x - 3) \)

Estimate \( h(3.1) \):
The gradient of \( f : \mathbb{R}^n \to \mathbb{R}^1 \) is denoted by

\[
\nabla f(a) = \left( \frac{\partial f}{\partial x_1}(a), \ldots, \frac{\partial f}{\partial x_n}(a) \right)
\]

Defn: The Jacobian matrix of \( f : \mathbb{R}^n \to \mathbb{R}^m \) at \( a \) is

\[
Df(a) = \left[ \frac{\partial f_i}{\partial x_j}(a) \right]_{m \times n} = \\
\begin{bmatrix}
\frac{\partial f_1}{\partial x_1}(a) & \ldots & \frac{\partial f_1}{\partial x_n}(a) \\
\vdots & \ddots & \vdots \\
\vdots & & \ddots \\
\frac{\partial f_m}{\partial x_1}(a) & \ldots & \frac{\partial f_m}{\partial x_n}(a)
\end{bmatrix}
\]

Thm: If \( f \) is differentiable at \( a \), then

1.) \( f \) is continuous at \( a \).

2.) \( \frac{\partial f_i}{\partial x_j} \) exists at \( a \) for all \( i, j \).

3.) The derivative of \( f \) at \( a = Df(a) = \) the Jacobian matrix of \( f \) at \( a \).

Thm: Let \( f : \mathbb{R}^n \to \mathbb{R}^m \), \( f = (f_1, \ldots, f_m) \). If \( \frac{\partial f_i}{\partial x_j} \) exists and are continuous in a neighborhood of \( a \) for all \( i, j \), then \( f \) is differentiable at \( a \).