1.1 Vectors:

Let \( \mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \).

If \( \mathbf{v} \) = velocity in m/sec of an object, then the object is moving east at a rate of 1 m/sec and north at a rate of 2 m/sec.

Speed of the object =

A vector can be described by its Euclidean coordinates OR by its length and direction.

Let \( \mathbf{w} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \). Then \( \mathbf{v} + \mathbf{w} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \).

\( \mathbf{v} - \mathbf{w} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \).

2.1 Let \( f : X \to Y \) where \( X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m \)

Graph of \( f = \{ (x, f(x)) \mid x \in X \} \subset \mathbb{R}^n \times \mathbb{R}^m \)

Domain of \( f = X \), Codomain of \( f = Y \)

Range of \( f = \text{Image of } f = f(X) = \{ y \in Y \mid \text{there exists } x \in X \text{ such that } f(x) = y \} \).

\( f \) is a function if for all \( x \) in domain of \( f \), \( f(x) \) has a unique value.

I.e., for all \( x, y \in X \), if \( x = y \), then \( f(x) = f(y) \) and for all \( x \in X \), \( f(x) \) is defined.

\( f \) is 1:1 if \( f(x) = f(y) \) implies \( x = y \).

\( f \) gives a one-to-one correspondence between \( X \) and \( f(X) \).

Given \( b \in Y \), \( f(x) = b \) has at most one solution

Side-note: \( f(x) = b \) has exactly one solution if \( b \in f(X) \).

Side-note: \( f(x) = b \) has no solution if \( b \notin f(X) \).

\( f \) is onto if \( f(X) = Y \) (i.e., image of \( f = \text{codomain of } f \)).

Given \( b \in Y \), \( f(x) = b \) has at least one solution.
Ex 1: $f : \mathbb{R}^n \to \mathbb{R}$, $f(x) = ||x|| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$

Domain = Codomain = Image =

Is $f$ 1:1?
Proof:

Is $f$ onto?
Proof:

Alternate Proof:

Ex 2: $g(x, y) = (x^2y, x^4 - y, x^6)$

Domain = Codomain =

Is $g$ 1:1?
Proof:

Is $g$ onto?
Proof:

Ex 3: $h(x) = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

I.e, $h(x) = (x + 2y + 3z, 4x + 5y + 6z)$.

Domain = Codomain = Image =

Is $h$ onto?
Is $h$ 1:1?

How many solutions does $h(x) = b$ have?

I.e., how many solutions does $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ have?

I.e, how many solutions does the following system of equations have:

$$x + 2y + 3z = b_1,$$
$$4x + 5y + 6z = b_2.$$  

Does $\begin{pmatrix} 1 \\ 4 \\ 2 \\ 5 \\ 3 \\ 6 \end{pmatrix}$ span all of $\mathbb{R}^2$?

Is $\{(\begin{pmatrix} 1 \\ 4 \\ 2 \\ 5 \\ 3 \\ 6 \end{pmatrix})\}$ linearly independent?
Definitions:

If the codomain of \( f \) is \( \mathbb{R} \) (i.e., \( f : X \to \mathbb{R} \)), we say that \( f \) is real-valued or scalar valued.

Suppose \( f : X \subset \mathbb{R}^2 \to \mathbb{R} \) and \( c \) is a constant scalar.

The \textit{level curve at height c of f} is the curve in \( \mathbb{R}^2 \) defined by \( f(x, y) = c \). That is,
the level curve at height \( c \) of \( f = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = c\} \).

The \textit{contour curve at height c of f} is the curve in \( \mathbb{R}^3 \) defined by the two equations, \( z = f(x, y), z = c \). That is,
the contour curve at height \( c \) of \( f = \{(x, y, z) \in \mathbb{R}^2 \mid z = f(x, y) = c\} \)
\( = \{(x, y, f(x, y)) \in \mathbb{R}^3 \mid f(x, y) = c\} \).

Recall the graph of \( f = \{(x, y, z) \mid z = f(x, y)\} \)
\( = \{(x, y, f(x, y)) \mid (x, y) \in X\} \subset \mathbb{R}^2 \times \mathbb{R} \)

The \textit{section of the graph of f by the plane x = c} is the set of points in \( \mathbb{R}^3 \) defined by the two equations, \( z = f(x, y), x = c \).
That is,
the section by \( x = c \) is \( \{(x, y, z) \in \mathbb{R}^2 \mid z = f(x, y), x = c\} \)
\( = \{(c, y, f(c, y)) \in \mathbb{R}^3 \mid (c, y) \in X\} \).

The section by \( y = c \) is \( \{(x, y, z) \in \mathbb{R}^2 \mid z = f(x, y), y = c\} \)
\( = \{(x, c, f(x, c)) \in \mathbb{R}^3 \mid (x, c) \in X\} \).