1a.) Use a linear approximation (or differentials) to estimate \( \ln(0.97) \)

Let \( f(x) = \ln(x) \). Note 0.97 is close to 1 and \( f(1) = \ln(1) = 0 \)

\[ f'(x) = \frac{1}{x} \] . Hence the slope of the tangent line to \( f \) at \( x = 1 \) is \( \frac{dy}{dx} = f'(1) = 1 \).

Since \( \Delta x = dx = 0.97 - 1 = -0.03, dy = f'(1)dx = 1(-0.03) = -0.03 \).

Thus \( \ln(0.97) = f(1) + \Delta y = 0 + \Delta y \sim dy = -0.03 \)

Alternate method: Since \( f(1) = 0 \) and \( f'(1) = 1 \), the equation of the tangent line to \( f(x) = \ln(x) \) at \( x = 1 \) is \( y = x - 1 \). Thus the tangent line \( L(x) = x - 1 \) is the linear approximation to \( f(x) = \ln(x) \) near \( x = 1 \). Since 0.97 is close to 1, \( f(x) \sim L(x) = 0.97 - 1 = -0.3 \).

Sidenote: In this case, the answer is a good approximation \( (\ln(0.97) = -0.0304592...) \), but whether or not a tangent line is a good approximation of a function near the point of tangency depends on the function.

1b.) Is the answer to 1a an over-estimate or an under-estimate? \textit{over-estimate}

\( f'(x) = x^{-1}, f''(x) = -x^{-2} < 0 \). Hence \( f \) is concave down and the answer is an over-estimate.

1c.) For the problem in 1a,

\[ dx = -0.03, dy = -0.03, \Delta x = -0.03, \Delta y = \ln(0.97). \]

2a.) The linearization of \( f(x) = \sin(x) \) at \( x = 0 \) is \( y = x \)

2b.) The linearization of \( f(x) = \cos(x) \) at \( x = 0 \) is \( y = 1 \)

2c.) The linearization of \( f(x) = \sin(x) \) at \( x = \frac{\pi}{2} \) is \( y = 1 \)

2d.) The linearization of \( f(x) = 2x + 1 \) at \( x = 0 \) is \( y = 2x + 1 \)

2e.) Use the above linearizations to estimate the following:

\[ \sin(0.1) \sim 0.1, \quad \sin\left(\frac{3\pi}{2}\right) \sim 1 \]
3.) Recall that radioactive substances decay at a rate proportional to the remaining mass. The half-life of polonium-218 is 3 minutes. Suppose a sample originally has a mass of 400g. SIMPLIFY your answers to the following:

a.) A formula for the mass remaining after \( t \) minutes is \( 400(2^{-\frac{t}{3}}) \)

b.) The mass remaining after 6 minutes is \( \frac{100}{400} = \frac{1}{4} \)

c.) When is the mass reduced to 10g? \( \log_2(64000) \)

Answer:

b.) If half life = 3 minutes and if \( m(0) = 400 \), then \( m(3) = 200 \) and \( m(6) = 100 \)

a.) radioactive substances decay at a rate proportional to the remaining mass: \( \frac{dm}{dt} = km \)

Note \( m(t) = m(0)e^{kt} \) is a solution to \( \frac{dm}{dt} = km \)

From ch 5: \( \int \frac{dm}{m} = \int kdt \). hence \( \ln|m| = kt + C \). Thus \( |m| = e^{kt+C} = e^{kt}e^C \). Since mass is always positive, we obtain \( m(t) = m(0)e^{kt} \)

\( 400e^{3t} = 200. \) Thus \( e^{3k} = \frac{1}{2} \). \( k = -\frac{\ln(2)}{3} \)

\( m(t) = 400e^{-\frac{\ln(2)}{3}t} = 400e^{\ln(2)\frac{-t}{3}} = 400(2^{-t/3}) \)

b.) \( m(6) = 400(2^{-6/3}) = 400(2^{-2}) = 400/4 = 100 \)

c.) \( 10 = 400(2^{-t/3}) \)

\( \frac{1}{40} = 2^{-t/3} \)

Taking the reciprocal of both sides: \( 40 = 2^{t/3} \)

\( \log_2(40) = \frac{t}{3} \)

\( t = 3\log_2(40) = \log_2(40^3) = \log_2(64000) \)
4.) Find the derivative of \( x \ln(\sqrt{3} \sin(x^2) - e^x + 1) \).
Circle your answer. You do NOT need to simplify.

\[
[x \cos(ln(\sqrt{3}e^x - x^2 + 1))]' = \\
\ln(\sqrt{3} \sin(x^2) - e^x + 1) + x\left(\frac{1}{\sqrt{3} \sin(x^2) - e^x + 1}\right)\left(\frac{1}{2}\right)\left(3\sin(x^2) - e^x + 1\right)^{-\frac{1}{2}}(6\cos(x^2) - e^x)
\]

5.) A plan flies horizontally at an altitude of 10km and passes directly over a tracking telescope on the ground. When the angle of elevation is \( \pi/3 \) (with respect to the tracking telescope, after it has passed over the tracking telescope), this angle is decreasing at a rate of \( \pi/4 \) rad/min. How fast is the plane traveling at that time. (Hint: you can use a right triangle).

\[\tan(\theta) = \frac{10}{x} \text{ Hence } \frac{\sin \theta}{\cos \theta} = \frac{10}{x} \text{. Note when } \theta = \pi/3, \theta' = -\pi/4\]

**method 1:** \( x \sin(\theta) = 10 \cos(\theta) \)

\[x' \sin(\theta) + x \cos(\theta) \theta' = -10 \sin(\theta) \theta' \]

when \( \theta = \pi/3, \theta' = -\pi/4 \). \( \tan(\pi/3) = \frac{10}{x} \) implies \( x = \frac{10}{\tan(\pi/3)} = \frac{10}{\sqrt{3}} \).

\[x' \sin(\pi/3) + \frac{10}{\sqrt{3}} \cos(\pi/3)(-\frac{\pi}{4}) = -10 \sin(\pi/3)(-\frac{\pi}{4}).\]

\[x' \frac{\sqrt{3}}{2} + \frac{10}{\sqrt{3}}(-\frac{\pi}{4}) = -10\left(\frac{\sqrt{3}}{2}\right)(-\frac{\pi}{4}).\]

\[x' \frac{\sqrt{3}}{2} = -10 \sqrt{3}(-\frac{\pi}{4}).\]

\[x' \frac{\sqrt{3}}{2} = -10 \sqrt{3}(-\frac{\pi}{4}) - \frac{10}{\sqrt{3}}(-\frac{\pi}{4}) = 10 \sqrt{3}(\frac{\pi}{4}) + \frac{10}{\sqrt{3}} (\frac{\pi}{4}).\]

\[x' = 10(\frac{\pi}{4}) + \frac{10}{3}(\frac{\pi}{4}) = \frac{40}{3}(\frac{\pi}{4}) = \frac{10\pi}{3}.\]

**method 2:** \( x = 10 \frac{\cos(\theta)}{\sin(\theta)} \)

\[x' = 10\left(-\frac{\sin(\theta) \theta' \sin(\theta) - \cos(\theta) \cos(\theta) \theta'}{\sin^2(\theta)}\right) = -10 \theta' \left(\frac{\sin^2(\theta) + \cos^2(\theta)}{\sin^2(\theta)}\right) = -\frac{10 \theta'}{\sin^2(\theta)}\]

when \( \theta = \pi/3, \theta' = -\pi/4 \), \( x' = \frac{10\pi/4}{\sin^2(\pi/3)} = \frac{10\pi/4}{(\sqrt{3}/2)^2} = \frac{10\pi/4}{3/4} = \frac{10\pi}{3} \)

Answer: \( \frac{10\pi}{3} \text{ km/min} \)
6.) Find the following for \( f(x) = x^3 - 8x^2 + 16x = (x - 4)^2 \) (if they exist; if they don’t exist, state so). Use this information to graph \( f \).

Note \( f'(x) = 3x^2 - 16x + 16 = (x - 4)(3x - 4) \) and \( f''(x) = 6x - 16 \)

[1.5] 6a.) critical numbers: \( \frac{4}{3}, 4 \)

[1.5] 6b.) local maximum(s) occur at \( x = \frac{4}{3} \)

[1.5] 6c.) local minimum(s) occur at \( x = 4 \)

[1.5] 6d.) The global maximum of \( f \) on the interval \([0, 5]\) is \( \frac{256}{27} \) and occurs at \( x = \frac{4}{3} \)

[1.5] 6e.) The global minimum of \( f \) on the interval \([0, 5]\) is \( 0 \) and occurs at \( x = 0, 4 \)

[1.5] 6f.) Inflection point(s) occur at \( x = \frac{8}{3} \)

[1.5] 6g.) \( f \) increasing on the intervals \( (-\infty, \frac{4}{3}) \cup (4, \infty) \)

[1.5] 6h.) \( f \) decreasing on the intervals \( (\frac{4}{3}, 4) \)

[1.5] 6i.) \( f \) is concave up on the intervals \( (\frac{8}{3}, \infty) \)

[1.5] 6j.) \( f \) is concave down on the intervals \( (-\infty, \frac{8}{3}) \)

[5] 6k.) Graph \( f \)
7.) Circle T for true and F for false. If the statement is false, give a counter-example.

7a.) If \( f'(x) > 0 \) on an interval, then \( f \) is increasing on that interval. \( T \)

Counter-example: \textit{None}.

7b.) If \( f \) is increasing on an interval, then \( f'(x) > 0 \) on that interval. \( F \)

Counter-example: \( f(x) = x^3 \) is an increasing function on \((-\infty, \infty)\), but \( f'(0) = 0 \).

7c.) If \( f'(c) = 0 \), then \( f \) has a local maximum or local minimum at \( c \). \( F \)

Counter-example: If \( f(x) = x^3 \), then \( f'(0) = 0 \), but \( f(0) \) is neither a local maximum nor a local minimum.

7d.) If \( f \) has a local maximum or local minimum at \( c \), then \( f'(c) = 0 \). \( F \)

Counter-example: \( f(x) = |x| \) has a local minimum at \( x = 0 \), but \( f'(0) \) does not exist.

7e.) If \( f \) has a local maximum or local minimum at \( c \) and if \( f'(c) \) exists, then \( f'(c) = 0 \). \( T \)

Counter-example: None

7f.) Suppose \( f'' \) is continuous near \( x \) and \( f'(c) = 0 \). If \( f''(c) > 0 \), then \( f \) has a local minimum at \( c \). \( T \)

Counter-example: None

7g.) Suppose \( f'' \) is continuous near \( x \) and \( f'(c) = 0 \). If \( f \) has a local minimum at \( c \), then \( f''(c) > 0 \). \( F \)

Counter-example: \( f(x) = x^4 \) has a local minimum at \( x = 0 \), but \( f''(0) = 0 \).

7h.) If \( f \) is continuous on \((a, b)\), then \( f \) attains an absolute maximum value \( f(c) \) at some number \( c \) in \((a, b)\). \( F \)

Counter-example: \( f : (0, 1) \rightarrow \mathbb{R}, f(x) = x \) does not have an absolute maximum value.