Final Exam PART A, Dec., 2005 SHOW ALL WORK
Math 25 Calculus I Either circle your answers or place on answer line.

All problems required on this part of the exam.

1.) Suppose $f(x) = 2x^2 - x + 1$ and $g(x) = x^2 + 2x + 5$.

$2x^2 - x + 1 = x^2 + 2x + 5$

$x^2 - 3x - 4 = 0$

$(x + 1)(x - 4) = 0$. Hence $x = -1, 4$

Take $0 \in (-1, 4)$

$x = 0 : 2x^2 - x + 1 = 1$

$x = 0 : x^2 + 2x + 5 = 5$

Hence $x^2 + 2x + 5 > 2x^2 - x + 1$ on (-1, 4)

[3] 1a.) Set up, **but do NOT evaluate**, an integral for the area of the region enclosed by f and g.

height $= x^2 + 2x + 5 - (2x^2 - x + 1) = -x^2 + 3x + 4$, width $= dx$

Area $= \int_{-1}^{4}(-x^2 + 3x + 4)dx$

[4] 1b.) Set up, **but do NOT evaluate**, an integral for the volume of the solid obtained by rotating the region bounded by the curves f and g about the line $x = 9$ (hint: use cylindrical shells).

Area $= 2\pi rh$, $h = x^2 + 2x + 5 - (2x^2 - x + 1) = -x^2 + 3x + 4$, $r = 9 - x$,

Area $= 2\pi(9 - x)(-x^2 + 3x + 4)$

width or thickness of cylindrical shell $= dx$.

Volume $= 2\pi \int_{-1}^{4}(9 - x)(-x^2 + 3x + 4)dx$

[4] 1c.) Set up, **but do NOT evaluate**, an integral for the volume of the solid obtained by rotating the region bounded by the curves f and g about the line $y = -2$ (hint: use washers).

Area $= \pi(R^2 - r^2)$, $R = x^2 + 2x + 5 - (-2) = x^2 + 2x + 7$, $r = 2x^2 - x + 1 - (-2) = 2x^2 - x + 3$

width or thickness of washer $= dx$.

Volume $= \pi \int_{-1}^{4}[(x^2 + 2x + 7)^2 - (2x^2 - x + 3)^2]dx$

[1] 2.) If $h(x) = x^2$, then the slope of the tangent line at the point (2, 4) is $\underline{4}$

$h'(x) = 2x$, $h'(2) = 2(2) = 4$

[10] 3.) Find the derivative of $g(x) = \sqrt{\frac{ln(sin(x))}{x+1}}$

Answer 3.) $\underline{\frac{1}{2}[\frac{ln(sin(x))}{x+1}]^{\frac{-1}{2}}[\frac{(x+1)(\frac{1}{sin(x)})cos(x)-ln(sin(x))(1)}{(x+1)^2}]}$

4.) Find the following integrals:

Note typo: the bottom should be $\sqrt{sin(x)+4}$

[10] a.) $\int_0^{\frac{\pi}{2}} \frac{cosx}{\sqrt{sin(x)+4}}dx = \underline{2\sqrt{5}-4}$

let $u = sin(x) + 4$

$du = cos(x)dx$

$x = 0 : u = sin(0) + 4 = 0 + 4 = 4$

$x = \frac{\pi}{2} : u = sin(\frac{\pi}{2}) + 4 = 1 + 4 = 5$

$\int_0^{\frac{\pi}{2}} \frac{cosx}{\sqrt{sin(x)+4}}dx = \int_4^5 \frac{du}{u^{\frac{1}{2}}} = \int_4^5 u^{\frac{-1}{2}}du = 2u^{\frac{1}{2}}|_4^5 = 2[\sqrt{5}-\sqrt{4}] = 2[\sqrt{5}-2] = 2\sqrt{5}-4$

[3] b.) $\int \frac{5}{1+x^2}dx = \underline{5tan^{-1}(x) + C}$ (from table)

Note typo: Should be right-handed limit.

[10] 5.) Find the following limit (SHOW ALL STEPS): $lim_{x\to0^+}(3x+1)^{\frac{1}{x^2}} = \underline{+\infty}$

$lim_{x\to0^+}(3x+1)^{\frac{1}{x^2}} = lim_{x\to0^+}e^{ln[(3x+1)^{\frac{1}{x^2}}]}$

$lim_{x\to0^+}ln[(3x+1)^{\frac{1}{x^2}}] = lim_{x\to0^+}\frac{1}{x^2}ln(3x+1) = lim_{x\to0^+}\frac{ln(3x+1)}{x^2}$ (" $\frac{0}{0}$ ")

$= lim_{x\to0^+}\frac{\frac{1}{3x+1}(3)}{2x}$ (by l'Hospital's rule)

$= lim_{x\to0^+}\frac{3}{(3x+1)2x} = +\infty$

$lim_{x\to0^+}(3x+1)^{\frac{1}{x^2}} = lim_{x\to0^+}e^{ln[(3x+1)^{\frac{1}{x^2}}]} = e^{lim_{x\to0^+}ln[(3x+1)^{\frac{1}{x^2}}]} = +\infty$

6.) Find the following for $f(x) = xe^{-2\sqrt{x}}$ (if they exist; if they don't exist, state so). Use this information to graph f.

Note $f'(x) = e^{-2\sqrt{x}}(1 - \sqrt{x})$ and $f''(x) = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}(\sqrt{x} - \frac{3}{2})$ and $lim_{x \to \infty} xe^{-2\sqrt{x}} = 0$

[1] 6a.) critical numbers: $\underline{x = 1}$

[1] 6b.) local maximum(s) occur at $x = \underline{x = 1}$

[1] 6c.) local minimum(s) occur at $x = \underline{none}$

[1] 6d.) The global maximum of f on the interval [0, 5] is $\underline{e^{-2}}$ and occurs at $x = \underline{1}$

[1] 6e.) The global minimum of f on the interval [0, 5] is $\underline{0}$ and occurs at $x = \underline{0}$ (since $f(0) = 0$, $f(1) = e^{-2}$, $f(5) = 5e^{-2\sqrt{5}}$

[1] 6f.) Inflection point(s) occur at $x = \underline{\frac{9}{4}}$

[1] 6g.) f increasing on the intervals $\underline{[0, 1)}$

[1] 6h.) f decreasing on the intervals $\underline{(1, \infty)}$

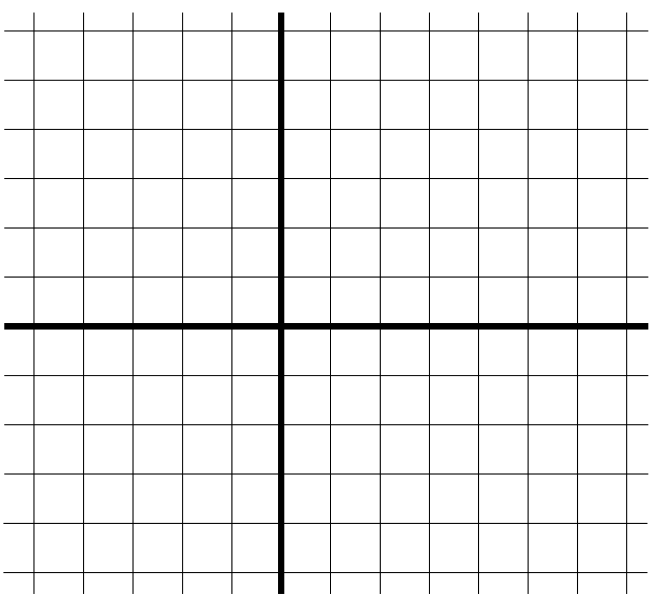
[1] 6i.) f is concave up on the intervals $\underline{[\frac{9}{4}, \infty)}$

[1] 6j.) f is concave down on the intervals$\underline{(0, \frac{9}{4})}$

[1] 6k.) What is the domain of f? $\underline{[0, \infty)}$

[1] 6l.) What is the range of f? $\underline{[0, e^{-2}]}$

[4] 6m.) Graph f

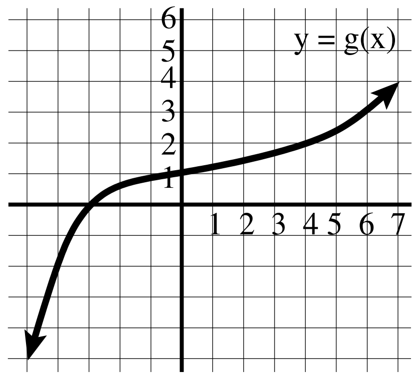


Final Exam PART B, Dec., 2005 SHOW ALL WORK
Math 25 Calculus I Either circle your answers or place on answer line.

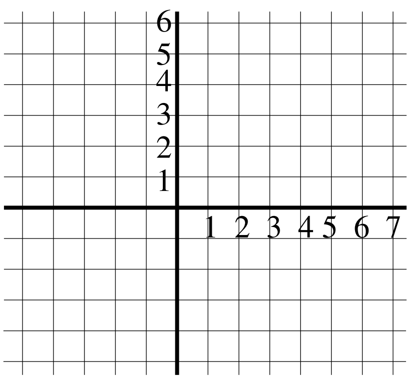
Choose 4 out of the following 5 problems: **Clearly indicate which 4 problems you choose.** Each problem is worth 10 points You may do all the problems for up to five points extra credit.

I have chosen the following 4 problems: ______________________________

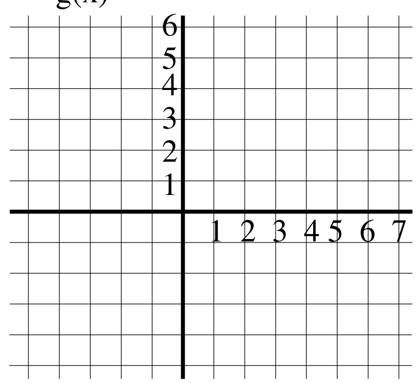
A.) Given the graph of $y = g(x)$ below, draw the following graphs:



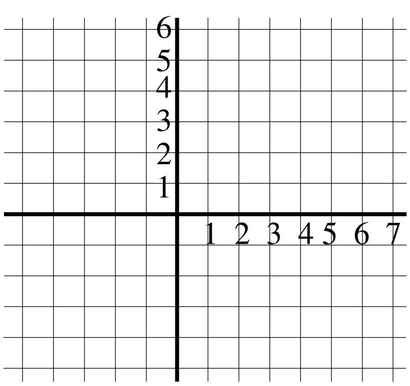
$y = g(x-1)$



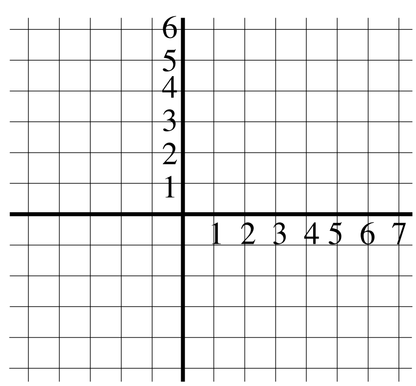
$y = \dfrac{1}{g(x)}$



$y = g^{-1}(x)$



$y = g'(x)$



B.) Use calculus to show that the equation $x^9 + 4x^3 + 10x = 0$ has at most one real root.

Let $f(x) = x^9 + 4x^3 + 10x$

Suppose $f(x)$ has two real roots. I.e, there exists a, b, $a \neq b$ such that $f(a) = 0 = f(b)$.

By the Mean Value Theorem (or Rolle's Thm), there exists c between a and b such that $f'(c) = \frac{f(b)-f(a)}{b-a} = 0$

Since $f(x) = x^9 + 4x^3 + 10x$, $f'(x) = 9x^8 + 12x^2 + 10$.

Thus $0 = f'(c) = 9c^8 + 12c^2 + 10$. But $9c^8 + 12c^2 + 10 > 0$.

Hence $f(x)$ has at most one real root.

[Note: could use the intermediate value thm (IVT) to show there is at least one real root, but the question didn't ask if there was a root. It only asked if there was more than one real root).

C.) Express the following integral as a limit of Riemann sums. Do not evaluate the limit: $\int_2^8 (x+1)sin(3x)dx$.

$\Delta x = \frac{8-2}{n} = \frac{6}{n} =$ width

$x_i = 2 + \frac{6i}{n}$. Thus $f(x_i) = (2 + \frac{6i}{n} + 1)sin(3(2 + \frac{6i}{n})) = (3 + \frac{6i}{n})sin(6 + \frac{18i}{n}) =$ height

Answer C.) $\underline{lim_{n\to\infty} \Sigma_{i=1}^{n} [(3 + \frac{6i}{n})sin(6 + \frac{18i}{n})] \frac{6}{n}}$

D.) Water is leaking out of an inverted conical tank at a rate of 30 m^3/sec at the same time that water is being pumped into the tank at a constant rate. The tank has height 10 m and the diameter at the top is 6 m. If the water level is rising at a rate of 4 m/sec when the height of the water is 5m, find the rate at which water is pumped into the tank.

$V = \frac{1}{3}\pi r^2 h$, $\frac{dV}{dt} = \frac{dV_{in}}{dt} - \frac{dV_{out}}{dt} = \frac{dV_{in}}{dt} - 30$

$\frac{dh}{dt} = 4$. When $h = 5$, $\frac{dV_{in}}{dt} =?$

$\frac{r}{h} = \frac{3}{10}$. Hence $r = \frac{3h}{10}$

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi (\frac{3h}{10})^2 h = \frac{3\pi h^3}{100}$.

Thus $\frac{dV}{dt} = \frac{9\pi}{100} h^2 \frac{dh}{dt} = \frac{9\pi}{100}(5)^2(4) = 9\pi$

Answer D.) $\underline{(9\pi + 30)m^3/sec}$

E.) Use calculus to find the point on the line $y = 2x + 8$ that is closest to the point (4, 3). Explain why your answer is optimal.

minimize $d(x) = \sqrt{(x-4)^2 + (2x+8-3)^2} = \sqrt{(x-4)^2 + (2x+5)^2}$

$d'(x) = \frac{1}{2}[(x-4)^2 + (2x+5)^2]^{\frac{-1}{2}}[2(x-4)+2(2x+5)(2)] = \frac{2(x-4)+2(2x+5)(2)}{2[(x-4)^2+(2x+5)^2]^{\frac{1}{2}}} = \frac{2x-8+8x+20}{2[(x-4)^2+(2x+5)^2]^{\frac{1}{2}}}$
$= \frac{10x+12}{2[(x-4)^2+(2x+5)^2]^{\frac{1}{2}}} =$

Hence $d'(x)$ exists everywhere and if $d'(x) = 0$, then $x = -\frac{12}{10}$

If $x < -\frac{12}{10}$, then $d'(x) < 0$. If $x > -\frac{12}{10}$, then $d'(x) > 0$. Hence d is a decreasing function on $(-\infty, -\frac{12}{10})$ and is an increasing function on $(-\frac{12}{10}, \infty)$. Thus the minimum distance occurs when $x = -\frac{12}{10}$

When $x = -\frac{12}{10}$, $y = 2(\frac{-12}{10}) + 8 = \frac{-12}{5} + \frac{40}{5} = \frac{28}{5}$

Answer E.) $\underline{(-\frac{12}{10}, \frac{28}{5})}$