[14] 1.) Given \( y = (x^2 + 1)^x \), find \( y' \). Simplify your answer.

\[
( x^2 + 1 )^x = e^{x \ln(x^2 + 1)} = e^{x \ln(x^2 + 1)}
\]

\[
y' = e^{x \ln(x^2 + 1)} \left[ x \left( \frac{1}{x^2 + 1} \right) (2x) + \ln(x^2 + 1) \right]
\]

\[
= (x^2 + 1)^x \left[ \frac{2x^2}{x^2 + 1} + \ln(x^2 + 1) \right]
\]

Answer 1.) \( (x^2 + 1)^{x-1} (2x^2) + (x^2 + 1)^x \ln(x^2 + 1) \)

[13] 2.) Given \( y x^2 + 10 = y^3 \), find \( y'' \). You do NOT need to simplify your answer and you can leave your answer in terms of \( x \) and \( y \) (and only in terms of \( x \) and \( y \), \( y' \) should not appear in your final answer).

\[
y'(2x) + y'(x^2) = 3y^2 y' \]

\[
\Rightarrow 2xy = (3y^2 - x^2) y' \Rightarrow y' = \frac{2xy}{3y^2 - x^2}
\]

\[
y'' = \frac{(2xy' + 2y)(3y^2 - x^2) - 2xy(6yy' - 2x)}{(3y^2 - x^2)^2}
\]

Answer 2.) \( y'' = \frac{(4x^2y + 2y)(3y^2 - x^2) - 2xy \left( \frac{12xy^2}{3y^2 - x^2} - 2x \right)}{(3y^2 - x^2)^2} \)
[14] 3.) Calculate the following limit. Show all steps.

\[ \lim_{x \to 0^+} x \ln(x) = \frac{0}{0} \]

\[ = \lim_{x \to 0^+} \ln(x) + \frac{x}{x-1} \]

\[ = \ln((x)^{-1}) \]

\[ = \frac{-1}{x-1} \ln(x) \]

\[ = \frac{-1}{x-1} \cdot \ln(x) \]

\[ = \frac{\ln(x)}{x-1} \]

\[ = \lim_{x \to 0^+} (-1) = 0 \]

[5] 4a.) State the Mean Value Theorem

If \( f \) is continuous on \([a, b]\) and \( f \) is differentiable on \((a, b)\), then there exists \( c \in (a, b) \) such that

\[ f'(c) = \frac{f(b) - f(a)}{b - a} \]

[4b.) Use the Mean Value Theorem (or Rolle's theorem) to show \( f(x) = \ln(x) + x \) is one-to-one [Hint: recall \( f \) is one-to-one if \( f(a) = f(b) \) implies \( a = b \). Assume \( f(a) = f(b) \) and show \( a = b \) WHEN \( a \) and \( b \) are in the domain of \( f \).

Suppose \( f(a) = f(b) \) and \( a, b \) are in the domain of \( f = (0, \infty) \).

Suppose \( a < b \)

Note \( f \) is cont on \([a, b]\) \& diff on \((a, b)\)

Hence by the MVT there exists \( c \in (a, b) \) such that

\[ f'(c) = \frac{f(b) - f(a)}{b - a} \]

\[ f(x) = \ln(x) + x \Rightarrow f'(x) = \frac{1}{x} + 1 \Rightarrow f'(c) = \frac{1}{c} + 1 \]

But \( c \in (0, \infty) \Rightarrow f'(c) = \frac{1}{c} + 1 > 0 \), contradicting \( f'(c) = 0 \)
5. Two people start at the same point, say the origin. Person A walks east at a constant rate of 1 m/s. Person B walks northeast (45 degrees north of east) at 2 m/s. What is the rate of change in the distance between person A and person B after 20 seconds [law of cosines: \( a^2 = b^2 + c^2 - 2bc \cos(\alpha) \)]

\[
\begin{align*}
C^2 &= A^2 + B^2 - 2AB \cos 45^\circ \\
\dot{C} &= \dot{AA'} + \dot{BB'} + \dot{C}\cos 45^\circ (AB' + A'B) \\
C' &= AA' + BB' + \cos 45^\circ (AB' + A'B) \\
C' &= \frac{(20)(1) + (40)(2) + \cos 45^\circ [(20)(2) + (1)(40)]}{\sqrt{2000 + 800\sqrt{2}}} \\
C &= \sqrt{2000 + 800\sqrt{2}}
\end{align*}
\]

Answer 5.

6. A box with a square base and open top must have volume of 1000 cm³. Find the dimensions of the box that minimizes the amount of material used.

\[
V = 1000 = x^2 y \quad \Rightarrow \quad y = \frac{1000}{x^2}
\]

\[
A = x^2 + 4xy = x^2 + 4x \left( \frac{1000}{x^2} \right) = \frac{4000}{x}
\]

\[
A'(x) = 2x - \frac{4000}{x^2} = 2x^3 - 4000
\]

\[
A'(x) = 0 \quad \Rightarrow \quad 2x^3 - 4000 = 0 \quad \Rightarrow \quad 2x^3 = 4000 \quad \Rightarrow \quad x = \sqrt[3]{2000}
\]

Answer 6.)\[\sqrt[3]{2000} \text{ cm} \times \sqrt[3]{2000} \text{ cm} \times \frac{1000}{(2000)^{1/3}} \text{ cm}\]
6.) Find the following for \( f(x) = \frac{4-x^2}{x^2-9} = \frac{(2-x)(2+x)}{(x-3)(x+3)} \) (if they exist; if they don’t exist, state so). Use this information to graph \( f \).

Note \( f'(x) = \frac{-10x}{(x^2-9)^2} \) and \( f''(x) = \frac{-30(x^2+3)}{(x^2-9)^3} \)

[1.5] 6a.) critical numbers: \( 0 \)

[1.5] 6b.) local maximum(s) occur at \( x = \) **none**

[1.5] 6c.) local minimum(s) occur at \( x = \) **0**

[1.5] 6d.) The global maximum of \( f \) on the interval \([0, 3]\) is **none** and occurs at \( x = \) __________

[1.5] 6e.) The global minimum of \( f \) on the interval \([0, 3]\) is **\(-\frac{4}{9}\)** and occurs at \( x = \)_**0**_

[1.5] 6f.) Inflection point(s) occur at \( x = \) **none**

[1.5] 6g.) \( f \) increasing on the intervals \( (0, 3) \cup (3, \infty) \)

[1.5] 6h.) \( f \) decreasing on the intervals \( (-\infty, -3) \cup (-3, 0) \)

[1.5] 6i.) \( f \) is concave up on the intervals \( (-3, 3) \)

[1.5] 6j.) \( f \) is concave down on the intervals \( (-\infty, -3) \cup (3, \infty) \)

[1.5] 6k.) Equation(s) of vertical asymptote(s) \( x = 3, \ x = -3 \)

[4] 6l.) Equation(s) of horizontal and/or slant asymptote(s) \( y = -1 \)

[4.5] 6m.) Graph \( f \)

\[
\lim_{x \to \pm \infty} \frac{4-x^2}{x^2-9} = \lim_{x \to \pm \infty} \frac{-2x}{2x} = \lim_{x \to \pm \infty} (\frac{1}{2}) = -1
\]

\[
\frac{x^2}{16-9} = \frac{-12}{7}
\]