Section 4.2

Mean Value Theorem: Suppose
1.) f continuous on $[a, b]$
2.) f differentiable on (a, b)

Then there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

Applications of MVT (including Rolle's)

Lemma: If $f'(x) = 0$ for all $x \in [a, b]$, then $f(x) = c$ for some constant c and for all $x \in [a, b]$.

Proof:

Claim: $f(x) = f(a)$ for all $x \in [a, b]$

Let $x \in [a, b]$.

Since f' exists on $[a, b]$, f is differentiable on $[a, b]$. Hence f is also continuous on $[a, b]$.

Thus by MVT, there exists $c \in (a, x)$ such that

$$f'(c) = \frac{f(x) - f(a)}{x - a}$$

Since $f'(c) = 0$, $0 = \frac{f(x)-f(a)}{x-a}$ and thus $0 = f(x) - f(a)$.

Hence $f(x) = f(a)$.

Cor: If $f'(x) = g'(x)$ for all $x \in [a, b]$, then $f(x) = g(x) + c$ for some constant c and for all $x \in [a, b]$.