Mean Value Theorem: Suppose
1.) f continuous on $[a, b]$
2.) f differentiable on (a, b)

Then there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

Ex 3: If $f'(x) = 0$ for all $x \in (a, b)$, then $f(x) = c$ for some constant c.

Proof: Take $x_0 \in (a, b)$

Show $f(x) = f(x_0)$ for all $x \in (a, b)$ [i.e., $c = f(x_0)$].

Without loss of generality, assume $x > x_0$.
(proof is similar when $x < x_0$).

f is continuous on $[x_0, x]$.
f is differentiable on (x_0, x).

By MVT, there exists $c \in (x_0, x)$ such that

$\frac{f(x)-f(x_0)}{x-x_0} = f'(c) = 0.$

Thus $f(x) - f(x_0) = 0$. Hence $f(x) = f(x_0)$.

Ex 4: If $f'(x) = g'(x)$ for all $x \in (a, b)$, then $f(x) = g(x) + c$ for some constant c.

Proof: $f'(x) = g'(x)$ implies $(f - g)'(x) = f'(x) - g'(x) = 0$.

Thus $(f - g)(x) = f(x) - g(x) = c$ for some constant c.
Thus $f(x) = g(x) + c$.

The Fundamental Theorem of Calculus: Suppose f continuous on $[a, b]$.

1.) If $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$.

2.) $\int_a^b f(t)dt = F(b) - F(a)$ where F is any antiderivative of f, that is $F' = f$.