Section 2.3

Theorem: If \( f(x) \leq g(x) \) near \( a \) (except possibly at \( a \)) and if \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} g(x) \) exist, then

\[
\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)
\]

Squeeze theorem:
If \( f(x) \leq g(x) \leq h(x) \) near \( a \) (except possibly at \( a \)) and if \( \lim_{x \to a} f(x) = L \) and \( \lim_{x \to a} h(x) = L \), then

\[
\lim_{x \to a} g(x) = L
\]

Example: \( g(x) = x \sin \frac{1}{x} \)

\[-|x| \leq x \sin \frac{1}{x} \leq |x|\]

\[
\lim_{x \to 0} (-|x|) = 0, \quad \lim_{x \to 0} (|x|) = 0.
\]

Hence, \( \lim_{x \to 0} (x \sin \frac{1}{x}) = 0 \)

Section 2.4

Informal Defn: \( \lim_{x \to a} f(x) = L \) if \( x \) close to \( a \) (except possibly at \( a \)) implies \( f(x) \) is close to \( L \).

Formal Defn: \( \lim_{x \to a} f(x) = L \) if

For all \( \epsilon > 0 \), there exists \( \delta > 0 \) such that \( 0 < |x - a| < \delta \) implies \( |f(x) - L| < \epsilon \).
**Formal Defn:** \( \lim_{x \to a} f(x) = L \) if

For all \( \epsilon > 0 \), there exists \( \delta > 0 \) such that

\[ 0 < |x - a| < \delta \implies |f(x) - L| < \epsilon. \]

**Proof:**

Let \( \epsilon > 0 \). Choose \( \delta = \boxed{\text{____}} \). Note \( \delta = \boxed{\text{____}} > 0 \).

Suppose \( 0 < |x - a| < \delta \).

Claim: \( |f(x) - L| < \epsilon. \)

**Defn:** \( \lim_{x \to a} f(x) = L \) if

for all \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that

\[ 0 < |x - a| < \delta \implies |f(x) - L| < \epsilon. \]

Show \( \lim_{x \to 1} 2 = \)

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Defn: \( \lim_{x \to a} f(x) = L \) if for all \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that \( 0 < |x - a| < \delta \) implies \( |f(x) - L| < \varepsilon \).

Show \( \lim_{x \to 4} 2x + 3 = \)

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Defn: \( \lim_{x \to a^-} f(x) = L \) if \( x \) close to \( a \) and \( x < a \) implies \( f(x) \) is close to \( L \).

Defn: \( \lim_{x \to a^+} f(x) = L \) if \( x \) close to \( a \) and \( x > a \) implies \( f(x) \) is close to \( L \).
Defn: \( \lim_{x \to a} f(x) = \infty \) if

\( x \) close to \( a \) (except possibly at \( a \))
implies \( f(x) \) is large.

Defn: \( \lim_{x \to a} f(x) = -\infty \) if

\( x \) close to \( a \) (except possibly at \( a \))
implies \( f(x) \) is negative and \( |f(x)| \) is large.

Section 2.5

Defn: \( f \) is continuous at \( a \) if \( \lim_{x \to a} f(x) = f(a) \)
(i.e., if \( \lim_{x \to a} f(x) = f(\lim_{x \to a} x) \))

Examples:

Ex: Polynomial, rational, root, trigonometric,
inverse trigonometric, exponential, logarithmic
functions are continuous functions.

Read left and right continuity
If \( f, g \) continuous at \( a, c \in \mathbb{R} \), then \( f + g, fg, cf, f/g \) (if \( g(a) \neq 0 \)) are continuous.

If \( g \) continuous at \( a \) and \( f \) continuous at \( g(a) \), then \( f \circ g \) continuous at \( a \).

Ex: \( \lim_{x \to 0} \frac{x^2-e^x}{\cos(x)} = \)

Intermediate value theorem: Suppose \( f \) continuous on \([a,b]\), \( f(a) \neq f(b) \) and \( n \) is between \( f(a) \) and \( f(b) \), then there exists \( c \in (a,b) \) such that \( f(c) = N \).

Example: Show that \( x^2 - 7x + 1 \) has a root between 0 and 1.

Section 2.3: To find vertical asymptotes, find all \( a \in \mathbb{R} \) such that 
\( lim_{x \to a^-} f(x) = \pm \infty \) and/or \( lim_{x \to a^+} f(x) = \pm \infty \)

Ex: \( f(x) = \frac{1}{(x+2)(x-3)^2} \)

Section 2.6: Horizontal asymptotes/limits at infinity

To find horizontal asymptotes: 
calculate \( lim_{x \to +\infty} f(x) \) and \( lim_{x \to -\infty} f(x) \)

IF \( lim_{x \to +\infty} f(x) = L \) where \( L \) is a finite real number, then \( y = L \) is a horizontal asymptote.

IF \( lim_{x \to -\infty} f(x) = K \) where \( K \) is a finite real number, then \( y = K \) is a horizontal asymptote.
Ex: \( f(x) = \frac{2x^3 - x^2 + 1}{8x^3 + x + 3} \)

\[ \lim_{x \to +\infty} \frac{2x^3 - x^2 + 1}{8x^3 + x + 3} = \]

Ex: \( f(x) = \frac{x^2 + 1}{2x^3 + x^2 - 3} \)

\[ \lim_{x \to +\infty} \frac{x^2 + 1}{2x^3 + x^2 - 3} = \]

Similarly, \( \lim_{x \to -\infty} \frac{x^2 + 1}{2x^3 + x^2 - 3} = \)

Horizontal asymptote(s):

Ex: \( f(x) = \frac{2x^5 + x^2 - 3}{x^2 + 1} \)

\[ \lim_{x \to +\infty} \frac{2x^5 + x^2 - 3}{x^2 + 1} = \]

Similarly, \( \lim_{x \to -\infty} \frac{2x^5 + x^2 - 3}{x^2 + 1} = \)

Horizontal asymptote(s):
Ex: $f(x) = \frac{2x}{\sqrt{x^2 + 1}}$

$$\lim_{x \to +\infty} \frac{2x}{\sqrt{x^2 + 1}} =$$

Ex: $f(x) = x^2 - x^3$

$$\lim_{x \to +\infty} x^2 - x^3 =$$

$$\lim_{x \to -\infty} x^2 - x^3 =$$

Horizontal asymptote(s):

Ex: $f(x) = x^{\frac{2}{3}} - x$

$$\lim_{x \to +\infty} x^{\frac{2}{3}} - x =$$

$$\lim_{x \to -\infty} x^{\frac{2}{3}} - x =$$

Horizontal asymptote(s):