Some definitions:

If F is an orientable surface in orientable 3-manifold M, then F has a collar neighborhood $F \times I \subset M$. F has two sides. Can push F (or portion of F) in one direction.

M is prime if every separating sphere bounds a ball.

M is irreducible if every sphere bounds a ball. M irreducible iff M prime or $M \cong S^2 \times S^1$.

A disjoint union of 2-spheres, S, is independent if no component of $M - S$ is homeomorphic to a punctured sphere (S^3 - disjoint union of balls).

F is properly embedded in M if $F \cap \partial M = \partial F$.

Two surfaces F_1 and F_2 are parallel in M if they are disjoint and $M - (F_1 \cup F_2)$ has a component X of the form $\overline{X} = F_1 \times I$ and $\partial \overline{X} = F_1 \cup F_2$.

A compressing disk for surface F in M^3 is a disk $D \subset M$ such that $D \cap F = \partial D$ and ∂D does not bound a disk in F (∂D is essential in F).

Defn: A surface $F^2 \subset M^3$ without S^2 or D^2 components is incompressible if for each disk $D \subset M$ with $D \cap F = \partial D$, there exists a disk $D' \subset F$ with $\partial D = \partial D'$

Defn: A ∂ compressing disk for surface F in M^3 is a disk $D \subset M$ such that $\partial D = \alpha \cup \beta$, $\alpha = D \cap F$, $\beta = D \cap \partial M$ and α is essential in F (i.e., α is not parallel to ∂F or equivalently $\nexists \gamma \subset \partial F$ such that $\alpha \cup \gamma = \partial D'$ for some disk $D' \subset F$.

Defn: If F has a ∂ compressing disk, then F is ∂ compressible. If F does not have a ∂ compressing disk, then F is ∂ incompressible.

Defn: essential = incompressible and ∂ incompressible.

Note: If M is irreducible with incompressible boundary, and if $A \subset M$ is an incompressible annulus, then A is ∂ parallel if and only if A is ∂ compressible.

Defn: An irreducible manifold M is atoroidal if every incompressible torus in M is boundary parallel.

Defn: A torus decomposition of M is a finite disjoint union $\mathcal{T}$ of incompressible tori contained in the interior of M s. t.

1.) Each component of $M|\mathcal{T}$ is either atoroidal or a SFS.

2.) $\mathcal{T}$ is minimal with respect to (1), i.e., no proper subcollection of tori in $\mathcal{T}$ satisfies (1).

JSJ decomposition theorem: Every compact irreducible 3 manifold has a torus decomposition $\mathcal{T}$ unique up to isotopy (note $\mathcal{T}$ may be empty).

Proof of JSJ decomposition:

Finiteness:

Lemma: A closed surface F in a closed 3-manifold with triangulation T can be isotoped so that F is transverse to all simplices of T and for all 3-simplices τ, each component of $F \cap \partial\tau$ is of the form:

Defn: F is a normal surface with respect to T if

1.) F is transverse to all simplices of T.

2.) For all 3-simplices τ, each component of $F \cap \partial\tau$ is of the form:

3.) Each component of $F \cap \tau$ is a disk.

Lemma 3.5: (1.) If F is a disjoint union of independent 2-spheres then F can be taken to be normal.

(2.) If F is a closed incompressible surface in a closed irreducible 3-manifold, then F can be taken to be normal.

Thm 3.6 (Haken) Let M be a compact irreducible 3-manifold. If S is a closed incompressible surface in M and no two components of S are parallel, then S has a finite number of components.

Uniqueness or JSJ decomposition:

Let $\mathcal{T}$ and $\mathcal{T}'$ be two torus decompositions of M. Isotope $\mathcal{T}'$ so that $\mathcal{T}$ intersects $\mathcal{T}$ transversely and so that $|\mathcal{T} \cap \mathcal{T}'|$ is minimal.

Claim $\mathcal{T} \cap \mathcal{T}' = \emptyset$.

Proof: Suppose there exists $T \in \mathcal{T}$ and $T' \in \mathcal{T}'$ such that $T \cap T' \neq \emptyset$.

$T \cap T' = \coprod$ essential s.c.c.

Let γ be a component of $T \cap T'$.

Let M_1, M_2 be the two components of $M|\mathcal{T}$ that meet T (note if T non-separating, then $M_1 = M_2$).

Let A_1, A_2 be the annuli components of $T'|\mathcal{T}$ that meet γ with $A_1 \subset M_1$, $A_2 \subset M_2$.

T' incompressible in M implies A_i incompressible in M_i.

If A_i boundary parallel in M_i, then can isotop T' to reduce $|\mathcal{T} \cap \mathcal{T}'|$, contradicting the minimality of $|\mathcal{T} \cap \mathcal{T}'|$.

Hence A_i is not boundary parallel in M_i and thus is ∂ incompressible in M_i. Therefore A_i is essential in M_i

By hypothesis M_i is atoroidal or SFS.

Lemma 1.16: If M_i is compact, connected, orientable, irreducible, atoroidal, and contains an essential annulus meeting only torus components of ∂M_i, then M_i is a SFS.

Hence M_i is SFS.

Lemma 1.11 (Waldhausen) F incompressible and ∂ incompressible $\subset M$ connected, compact, irreducible SFS implies F isotopic to vertical ($=$ union of regular fibers $= p^{-1}(scc)=$ torus or Klein bottle or $p^{-1}(arc)$ $=$ annulus) or horizontal surface ($=$ surface transverse to all fibers)

Lemma 1.14: An incompressible and ∂ incompressible annulus in a compact SFS can be isotoped to be vertical, after possible changing the Seifert fibering.

Hence, can assume A_i vertical in M_i

Hence γ is a fiber in the Seifert fibration of M_i. Therefore $M_1 \cup_T M_2$ is SFS.

Hence can discard T from $\mathcal{T}$ contradicting the minimality of $\mathcal{T}$.

Hence $\mathcal{T} \cap \mathcal{T}' = \emptyset$.

If $\mathcal{T} = \emptyset$, then $\mathcal{T}' = \emptyset$.

So assume $\mathcal{T} \cap \mathcal{T}' = \emptyset$ and $\mathcal{T}, \mathcal{T}' \neq \emptyset$.

Suppose there exists $T \in \mathcal{T}$ such that T is boundary parallel in $M|\mathcal{T}'$. Then there exists $T' \in \mathcal{T}'$ such that T is parallel to T'. Hence can isotope T to T'. $\mathcal{T} - T$ and $\mathcal{T}' - T'$ are torus decompositions of $M - T$. Continue removing tori while there exist boundary parallel tori.

Suppose there exists a T which is not boundary parallel in $M|\mathcal{T}'$.

Let Q be a component of $M|\mathcal{T} \cup \mathcal{T}'$. such that $T \subset Q$ and $Q \cap \mathcal{T}' \neq \emptyset$.

Let M_1' be the component of $M|\mathcal{T}'$ containing Q.

Since T is not boundary parallel in $M|\mathcal{T}'$, T is not boundary parallel in M_1'.

Hence T is incompressible and not boundary parallel in M_1'.

Thus M_1' is not atoroidal. Hence M_1' is a SFS.

Lemma 1.15' If M is SFS with $|\partial M| \geq 2$ and $M \neq T^2 \times I$, then if ϕ and ϕ' are any two Seifert fiberings on M, then $\phi_{\partial M} = \phi'_{\partial M}$.