Some definitions:

If $F$ is an orientable surface in orientable 3-manifold $M$, then $F$ has a collar neighborhood $F \times I \subset M$. $F$ has two sides. Can push $F$ (or portion of $F$) in one direction.

$M$ is prime if every separating sphere bounds a ball.

$M$ is irreducible if every sphere bounds a ball. $M$ irreducible iff $M$ prime or $M \cong S^2 \times S^1$.

A disjoint union of 2-spheres, $S$, is independent if no component of $M - S$ is homeomorphic to a punctured sphere ($S^3$ - disjoint union of balls).

$F$ is properly embedded in $M$ if $F \cap \partial M = \partial F$.

Two surfaces $F_1$ and $F_2$ are parallel in $M$ if they are disjoint and $M - (F_1 \cup F_2)$ has a component $X$ of the form $\overline{X} = F_1 \times I$ and $\partial \overline{X} = F_1 \cup F_2$.

A compressing disk for surface $F$ in $M^3$ is a disk $D \subset M$ such that $D \cap F = \partial D$ and $\partial D$ does not bound a disk in $F$ ($\partial D$ is essential in $F$).

Defn: A surface $F^2 \subset M^3$ without $S^2$ or $D^2$ components is incompressible if for each disk $D \subset M$ with $D \cap F = \partial D$, there exists a disk $D' \subset F$ with $\partial D = \partial D'$.
Defn: A \( \partial \) compressing disk for surface \( F \) in \( M^3 \) is a disk \( D \subset M \) such that \( \partial D = \alpha \cup \beta \), \( \alpha = D \cap F \), \( \beta = D \cap \partial M \) and \( \alpha \) is essential in \( F \) (i.e., \( \alpha \) is not parallel to \( \partial F \) or equivalently \( \not\exists \gamma \subset \partial F \) such that \( \alpha \cup \gamma = \partial D' \) for some disk \( D' \subset F \).

Defn: If \( F \) has a \( \partial \) compressing disk, then \( F \) is \( \partial \) compressible. If \( F \) does not have a \( \partial \) compressing disk, then \( F \) is \( \partial \) incompressible.

Defn: essential = incompressible and \( \partial \) incompressible.

Note: If \( M \) is irreducible with incompressible boundary, and if \( A \subset M \) is an incompressible annulus, then \( A \) is \( \partial \) parallel if and only if \( A \) is \( \partial \) compressible.

Defn: An irreducible manifold \( M \) is atoroidal if every incompressible torus in \( M \) is boundary parallel.

Defn: A torus decomposition of \( M \) is a finite disjoint union \( \mathcal{T} \) of incompressible tori contained in the interior of \( M \) s. t.
1.) Each component of \( M|\mathcal{T} \) is either atoroidal or a SFS.
2.) \( \mathcal{T} \) is minimal with respect to (1), i.e., no proper subcollection of tori in \( \mathcal{T} \) satisfies (1).

JSJ decomposition theorem: Every compact irreducible 3 manifold has a torus decomposition \( \mathcal{T} \) unique up to isotopy (note \( \mathcal{T} \) may be empty).
Proof of JSJ decomposition:
Finiteness:
Lemma: A closed surface $F$ in a closed 3-manifold with tri-
angulation $T$ can be isotoped so that $F$ is transverse to all simplices of $T$ and for all 3-simplices $\tau$, each component of $F \cap \partial \tau$ is of the form:

Defn: $F$ is a normal surface with respect to $T$ if
1.) $F$ is transverse to all simplices of $T$.
2.) For all 3-simplices $\tau$, each component of $F \cap \partial \tau$ is of the form:
3.) Each component of $F \cap \tau$ is a disk.

Lemma 3.5: (1.) If $F$ is a disjoint union of independent 2-
spheres then $F$ can be taken to be normal.
(2.) If $F$ is a closed incompressible surface in a closed irre-
ducible 3-manifold, then $F$ can be taken to be normal.

Thm 3.6 (Haken) Let $M$ be a compact irreducible 3-manifold. If $S$ is a closed incompressible surface in $M$ and no two com-
ponents of $S$ are parallel, then $S$ has a finite number of com-
ponents.
Uniqueness or JSJ decomposition:

Let $\mathcal{T}$ and $\mathcal{T}'$ be two torus decompositions of $M$. Isotope $\mathcal{T}'$ so that $\mathcal{T}$ intersects $\mathcal{T}$ transversely and so that $|\mathcal{T} \cap \mathcal{T}'|$ is minimal.

Claim $\mathcal{T} \cap \mathcal{T}' = \emptyset$.

Proof: Suppose there exists $T \in \mathcal{T}$ and $T' \in \mathcal{T}'$ such that $T \cap T' \neq \emptyset$.

$T \cap T' = \bigsqcup$ essential s.c.c.

Let $\gamma$ be a component of $T \cap T'$.

Let $M_1, M_2$ be the two components of $M|\mathcal{T}$ that meet $T$ (note if $T$ non-separating, then $M_1 = M_2$).

Let $A_1, A_2$ be the annuli components of $T'|\mathcal{T}$ that meet $\gamma$ with $A_1 \subset M_1, A_2 \subset M_2$.

$T'$ incompressible in $M$ implies $A_i$ incompressible in $M_i$.

If $A_i$ boundary parallel in $M_i$, then can isotop $T'$ to reduce $|\mathcal{T} \cap \mathcal{T}'|$, contradicting the minimality of $|\mathcal{T} \cap \mathcal{T}'|$.

Hence $A_i$ is not boundary parallel in $M_i$ and thus is $\partial$ incompressible in $M_i$. Therefore $A_i$ is essential in $M_i$.

By hypothesis $M_i$ is atoroidal or SFS.
Lemma 1.16: If $M_i$ is compact, connected, orientable, irreducible, atoroidal, and contains an essential annulus meeting only torus components of $\partial M_i$, then $M_i$ is a SFS.

Hence $M_i$ is SFS.

Lemma 1.11 (Waldhausen) $F$ incompressible and $\partial$ incompressible $\subset M$ connected, compact, irreducible SFS implies $F$ isotopic to vertical (= union of regular fibers $= p^{-1}(scc)=$ torus or Klein bottle or $p^{-1}(arc) = \text{annulus}$) or horizontal surface (= surface transverse to all fibers)

Lemma 1.14: An incompressible and $\partial$ incompressible annulus in a compact SFS can be isotoped to be vertical, after possible changing the Seifert fibering.

Hence, can assume $A_i$ vertical in $M_i$

Hence $\gamma$ is a fiber in the Seifert fibration of $M_i$. Therefore $M_1 \cup_T M_2$ is SFS.

Hence can discard $T$ from $\mathcal{T}$ contradicting the minimality of $\mathcal{T}$.

Hence $\mathcal{T} \cap \mathcal{T}' = \emptyset$.

If $\mathcal{T} = \emptyset$, then $\mathcal{T}' = \emptyset$. 
So assume $T \cap T' = \emptyset$ and $T, T' \neq \emptyset$.

Suppose there exists $T \in \mathcal{T}$ such that $T$ is boundary parallel in $M|\mathcal{T}'$. Then there exists $T' \in \mathcal{T}'$ such that $T$ is parallel to $T'$. Hence can isotope $T$ to $T'$. $\mathcal{T} - T$ and $\mathcal{T}' - T'$ are torus decompositions of $M - T$. Continue removing tori while there exist boundary parallel tori.

Suppose there exists a $T$ which is not boundary parallel in $M|\mathcal{T}'$.

Let $Q$ be a component of $M|\mathcal{T} \cup \mathcal{T}'$. such that $T \subset Q$ and $Q \cap \mathcal{T}' \neq \emptyset$.

Let $M'_1$ be the component of $M|\mathcal{T}'$ containing $Q$.

Since $T$ is not boundary parallel in $M|\mathcal{T}'$, $T$ is not boundary parallel in $M'_1$.

Hence $T$ is incompressible and not boundary parallel in $M'_1$.

Thus $M'_1$ is not atoroidal. Hence $M'_1$ is a SFS.

Lemma 1.15' If $M$ is SFS with $|\partial M| \geq 2$ and $M \neq T^2 \times I$, then if $\phi$ and $\phi'$ are any two Seifert fiberings on $M$, then $\phi_{\partial M} = \phi'_{\partial M}$.