Defn: $M$ is an $n$-dimensional manifold (with boundary) if

1.) For all $x \in M$, there exists a neighborhood $V_x$ such that $V_x$ is homeomorphic to an open set in $R^n$ or $R_+^n$

2.) $M$ is $T_2$ and ...

Give an example of a topological space which satisfies (1), but is not $T_2$.

Answer: Friday’s Lecture.

$M$ is a closed manifold if $M$ is a compact manifold without boundary.

Wild knot:

Alexander horned sphere: see handout

To avoid such pathologies, we will work in the differentiable ($C^\infty$) or piecewise linear (PL) category.
Examples of n-manifolds:

\[ D^n = B^n = \{ x \in \mathbb{R}^n \mid ||x|| \leq 1 \} \]

\[ S^n = \{ x \in \mathbb{R}^{n+1} \mid ||x|| = 1 \} = \partial B^{n+1} \]

\[ P^n = S^n/(x \sim -x) \]

\[ T^n = S^1 \times S^1 \times ... \times S^1 \]

Forming new manifolds from old manifolds:

If \( M \) is an \( m \)-manifold and \( N \) is an \( n \)-manifold, then \( M \times N \) is a \((m+n)\)-manifold.

If \( M \) is an \( m \)-manifold, then \( \partial M \) is an \((m-1)\)-manifold.

Suppose \( M \) and \( N \) are \( n \)-manifolds and \( f \) : a component of \( \partial M \rightarrow \) a component of \( \partial N \) is a homeomorphism, then

\[ M \cup_f N = M \cup N/(x \sim f(x)) \]

In particular, \( M \# N = (M - B^n) \cup_i (N - B^n) \) where \( i : S^{n-1} \rightarrow S^{n-1} \).
\[ F_g = \# T^2 = (S^2 - \cup_{i=1}^{2g} D^2) \cup (\cup_{i=1}^{g} A^2) \] where \( A^2 = \) annulus

\[ N_g = \# P^2 = (S^2 - \cup_{i=1}^{g} D^2) \cup (\cup_{i=1}^{g} V^2) \] where \( V^2 = \) mobius band

Euler characteristic = \( \chi(M) = \) vertices - edges + faces - ... 
\[ = \Sigma_{i=0}^{\infty} (-1)^i \alpha_i(M) \] where \( \alpha_i(M) = \) number of \( i \) cells.
\[ = \Sigma_{i=0}^{\infty} (-1)^i \beta_i(M) \] where \( \beta_i(M) = dim H_i(M) \)

\[ \chi(M_1 \cup_F M_2) = \chi(M_1) + \chi(M_2) - \chi(F) \]

\[ \chi(S^{2n-1}) = 0. \quad \chi(S^{2n}) = 2. \quad \chi(D^n) = 1. \]

If \( S \) is a surface (compact connected 2-manifold) consisting of disjoint disks with bands attached, then
\[ \chi(S) = \# \text{ of disks} - \# \text{ of bands}. \]

\[ \chi(T^2) = 0. \quad \chi(T^2 \# T^2) = -2, \quad \chi(F_g) = 2 - 2g \]
\[ \chi(P^2) = 1. \quad \chi(P^2 \# P^2) = 0, \quad \chi(N_g) = 2 - g \]

Casson: "For three-dimensional topology, intuitive understanding is much more important than technical details."