Biology application: Suppose the number of bacteria grow at an average rate of $r = 10\%$ per hour. If the initial population is 100 bacteria, find the number of bacteria after 1 hour, after 2 hours, after $T$ hours.

Identical application, but in Finance:

Let $P(t) = \text{amount in an account at time } t \text{ (in years)}$.

Ex 1: Suppose $100$ is deposited in the account earning an interest rate of $r = 10\%$ per year. Find the amount in the account after 1 year, after 2 years, after 3 years, and after $T$ years.

$t = 0$: $P(0) = 100$

$t = 1$: $P(1) = 100(1 + 0.1) = 100(1.1) = 110$

$t = 2$: $P(2) = 100(1 + 0.1)(1 + 0.1) = 100(1 + 0.1)^2 = 100(1.1)^2 = 121$

$t = 3$: $P(3) = 100(1 + 0.1)^3 = 133.10$

$t = T$: $P(T) = 100(1 + 0.1)^T = 100(1.1)^T$

The average interest rate earned is $10\%$ per year.

The average rate of change in the account between year 0 and year 1:

$$\frac{P(1) - P(0)}{1} = 100(1.1) - 100 = 100(0.1) = 10 \text{/year}.$$ 

The average rate of change between year $t$ and year $t + 1$:

$$\frac{P(t+1) - P(t)}{1} = 100(1.1)^{t+1} - 100(1.1)^t = 100(1.1)^t(0.1) \text{/year}.$$ 

Instantaneous rate of change at time $t$:

$$P'(t) = [100(1.1)^t]' = 100ln(1.1)(1.1)^t = (9.53102...) \cdot (1.1)^t$$

At $t = 1$: $P'(1) = 100ln(1.1)(1.1) = 10.48...$
Ex 2: Suppose $100 is deposited in the account earning an interest rate of $r = 10\%$ per year, compounded monthly. Find the amount in the account after 1 month, 1 year, 2 years, and $T$ years.

$t = 0: P(0) = $100

$t = 1$ month: $P(\frac{1}{12}) = 100(1 + \frac{0.1}{12}) = $100.83

$t = 1$ year: $P(1) = 100(1 + \frac{0.1}{12})^{12} = $110.47

$t = 2$ years: $P(2) = 100(1 + \frac{0.1}{12})^{12 \cdot 2} = $122.04

$t = T$ years: $P(T) = 100(1 + \frac{0.1}{12})^{12 \cdot T} = $100(1.1047...)^T

The average interest rate earned is $\frac{10}{12}\%$ per month.
The average interest rate earned is $10.47...\%$ per year.
The average rate of change between year $t$ and year $t + 1$:

$$\frac{P(t+1) - P(t)}{t} = 100(1 + \frac{0.1}{12})^{12(t+1)} - 100(1 + \frac{0.1}{12})^{12t}$$

$$= $100(1 + \frac{0.1}{12})^{12t}[(1 + \frac{0.1}{12})^{12} - 1]/year.$$

The approximate average rate of change between year $t$ and year $t + 1$:

$$\frac{P(t+1) - P(t)}{t} = 100(1.1047)^{t+1} - 100(1.1047)^t$$

$$= $100(1.1047)^t(0.1047)/year.$$

The instantaneous rate of change at time $t$:

$$P'(t) = [100(1 + \frac{0.1}{12})^{12 \cdot t}]' = 100ln[(1 + \frac{0.1}{12})^{12}] \cdot [(1 + \frac{0.1}{12})^{12}]^t$$

$$= 1200ln(1 + \frac{0.1}{12}) \cdot (1 + \frac{0.1}{12})^{12t}$$

$$= (9.95856...) \cdot (1 + \frac{0.1}{12})^{12t}$$

At $t = 1 : P'(1) = 1200ln(1 + \frac{0.1}{12}) \cdot (1 + \frac{0.1}{12})^{12} = 11.001...$

At $t = \frac{1}{12}, P'(\frac{1}{12}) = 1200ln(1 + \frac{0.1}{12}) \cdot (1 + \frac{0.1}{12}) = 10.0416$
Ex 3: Suppose $100 is deposited in the account earning an interest rate of $r = 10\%$ per year, compounded daily. Find the amount in the account after 1 day, 1 year, 2 years, and $T$ years.

$t = 0$: $P(0) = $100

$t = 1$ day: $P(\frac{1}{365}) = 100(1 + \frac{0.1}{365}) = $100.03

$t = 1$ year: $P(1) = 100(1 + \frac{0.1}{365})^{365} = $110.52

$t = 2$ years: $P(2) = 100(1 + \frac{0.1}{365})^{365\cdot2} = $122.14

$t = T$ years: $P(T) = 100(1 + \frac{0.1}{365})^{365\cdot T} = $100(1.10515578...)^{T}$

The average interest rate earned is $\frac{10}{365} \%$ per day.

The average interest rate earned is $10.515578\ldots \%$ per year.

The average rate of change between year $t$ and year $t + 1$:

$$\frac{P(t+1) - P(t)}{1} = 100(1 + \frac{0.1}{365})^{365(t+1)} - 100(1 + \frac{0.1}{365})^{365t}$$

$$= $100(1 + \frac{0.1}{365})^{365t}[(1 + \frac{0.1}{365})^{365} - 1]/\text{year}.$$

The instantaneous rate of change at time $t$:

$$P'(t) = [100(1 + \frac{0.1}{365})^{365\cdot t}]' = 100ln[(1 + \frac{0.1}{365})^{365}] \cdot [(1 + \frac{0.1}{365})^{365}]^t$$

$$= 36500ln(1 + \frac{0.1}{365}) \cdot (1 + \frac{0.1}{365})^{365t}$$

$$= (9.99863\ldots) \cdot (1 + \frac{0.1}{365})^{365t}$$

At $t = 1$ : $P'(1) = 36500ln(1 + \frac{0.1}{365}) \cdot (1 + \frac{0.1}{365})^{365} = 11.05\ldots$

At $t = \frac{1}{365}$, $P'(\frac{1}{365}) = 36500ln(1 + \frac{0.1}{365}) \cdot (1 + \frac{0.1}{365}) = 10.00\ldots$
Ex 4: Suppose $100 is deposited in the account earning an interest rate of $r = 10\%$ per year, compounded $n$ times per year. Find the amount in the account after $T$ years.

$t = T$ years: $P(T) = 100(1 + \frac{0.1}{n})^{n \cdot T}$

Ex 5: Suppose $100 is deposited in the account earning an interest rate of $r = 10\%$ per year, compounded continuously. Find the amount in the account after $T$ years.

$t = T$ years: $P(T) = \lim_{n \to \infty} 100(1 + \frac{0.1}{n})^{n \cdot T} = 100e^{0.1T}$

Definition $e = \lim_{n \to \infty} (1 + \frac{1}{n})^n = 2.7...$

FYI : By Taylor series approximation from Calculus II

$e^{0.1T} = \lim_{n \to \infty}(1 + \frac{1}{n})^{n(0.1)T} = \lim_{n \to \infty}[(1 + \frac{1}{n})^{0.1}]^{nT}$

$= \lim_{n \to \infty}[1 + \frac{0.1}{n} - \frac{0.045}{n^2} + \frac{0.0285}{n^3} - ...]^{nT} = \lim_{n \to \infty}[1 + \frac{0.1}{n}]^{nT}$

Ex 6: Suppose $P_0$ is deposited in the account earning an interest rate of $r = s\%$ per year ($r = \frac{s}{100}$), compounded continuously.

$t$ years: $P(t) = \lim_{n \to \infty} P_0(1 + \frac{r}{n})^{n \cdot t} = P_0 e^{rt}$

Ex 7: Suppose $\$1 is deposited in the account earning an interest rate of $r = 10\%$ per year ($r = \frac{100}{100} = 1$), compounded continuously.

$t$ years: $P(t) = \lim_{n \to \infty} (1 + \frac{1}{n})^{n \cdot t} = e^{0.1t}$

Note the instantaneous rate of change is $10\% = 0.1e^{0.1t}$

That is $P'(t) = [e^{0.1t}]' = 0.1e^{0.1t}$

Ex 8: Suppose $\$1 is deposited in the account earning an interest rate of $r = 100\%$ per year ($r = \frac{100}{100} = 1$), compounded continuously.

$t$ years: $P(t) = \lim_{n \to \infty} (1 + \frac{1}{n})^{n \cdot t} = e^t$

Note the instantaneous rate of change is $100\% = e^t$

That is $P'(t) = [e^t]' = e^t$