

## Boyce/DiPrima 9<sup>th</sup> ed, Ch 2.7: Numerical Approximations: Euler's Method

Elementary Differential Equations and Boundary Value Problems, 9<sup>th</sup> edition, by William E. Boyce and Richard C. DiPrima, ©2009 by John Wiley & Sons, Inc.

- ✱ Recall that a first order initial value problem has the form

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

- ✱ If  $f$  and  $\partial f/\partial y$  are continuous, then this IVP has a unique solution  $y = \phi(t)$  in some interval about  $t_0$ .
- ✱ When the differential equation is linear, separable or exact, we can find the solution by symbolic manipulations.
- ✱ However, the solutions for most differential equations of this form cannot be found by analytical means.
- ✱ Therefore it is important to be able to approach the problem in other ways.

## Numerical Methods

- ✱ For our first order initial value problem

$$y' = f(t, y), \quad y(t_0) = y_0,$$

- ✱ an alternative is to compute approximate values of the solution  $y = \phi(t)$  at a selected set of  $t$ -values.
- ✱ Ideally, the approximate solution values will be accompanied by error bounds that ensure the level of accuracy.
- ✱ There are many numerical methods that produce numerical approximations to solutions of differential equations, some of which are discussed in Chapter 8.
- ✱ In this section, we examine the **tangent line method**, which is also called **Euler's Method**.

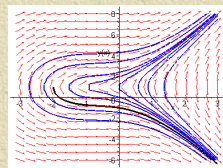
## General Error Analysis Discussion (1 of 4)

- ✱ Recall that if  $f$  and  $\partial f/\partial y$  are continuous, then our first order initial value problem

$$y' = f(t, y), \quad y(t_0) = y_0$$

has a solution  $y = \phi(t)$  in some interval about  $t_0$ .

- ✱ In fact, the equation has infinitely many solutions, each one indexed by a constant  $c$  determined by the initial condition.
- ✱ Thus  $\phi$  is the member of an infinite family of solutions that satisfies  $\phi(t_0) = y_0$ .



Check to see if a solution exists; Otherwise you might find a solution which doesn't actually exist.

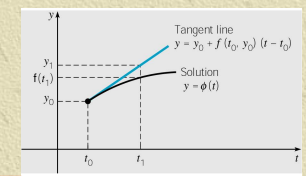
## Euler's Method: Tangent Line Approximation

- ✱ For the initial value problem

$$y' = f(t, y), \quad y(t_0) = y_0,$$

we begin by approximating solution  $y = \phi(t)$  at initial point  $t_0$ .

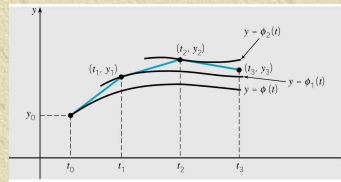
- ✱ The solution passes through initial point  $(t_0, y_0)$  with slope  $f(t_0, y_0)$ . The line tangent to the solution at this initial point is  $y = y_0 + f(t_0, y_0)(t - t_0)$
- ✱ The tangent line is a good approximation to solution curve on an interval short enough.
- ✱ Thus if  $t_1$  is close enough to  $t_0$ , we can approximate  $\phi(t_1)$  by  $y_1 = y_0 + f(t_0, y_0)(t_1 - t_0)$



3.6: linearization.

## General Error Analysis Discussion (2 of 4)

- ✦ The first step of Euler's method uses the tangent line to  $\phi$  at the point  $(t_0, y_0)$  in order to estimate  $\phi(t_1)$  with  $y_1$ .
- ✦ The point  $(t_1, y_1)$  is typically not on the graph of  $\phi$ , because  $y_1$  is an approximation of  $\phi(t_1)$ .
- ✦ Thus the next iteration of Euler's method does not use a tangent line approximation to  $\phi$ , but rather to a nearby solution  $\phi_1$  that passes through the point  $(t_1, y_1)$ .
- ✦ Thus Euler's method uses a succession of tangent lines to a sequence of different solutions  $\phi, \phi_1, \phi_2, \dots$  of the differential equation.



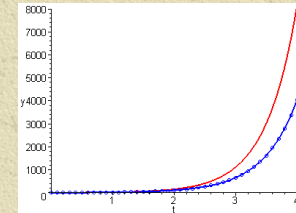
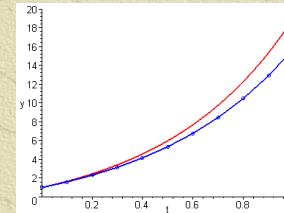
## Example 3: Error Analysis & Graphs (3 of 3)

- ✦ Given below are graphs showing the exact solution (red) plotted together with the Euler approximation (blue).

t	Exact y	Approx y	Error	% Rel Error
0.00	1.00	1.00	0.00	0.00
1.00	19.07	15.78	3.29	17.27
2.00	149.39	104.68	44.72	29.93
3.00	1109.18	652.53	456.64	41.17
4.00	8197.88	4042.12	4155.76	50.69

Exact Solution :

$$y = -\frac{7}{4} + \frac{1}{2}t + \frac{11}{4}e^{2t}$$



## Euler's Formula

- ✦ For a point  $t_2$  close to  $t_1$ , we approximate  $\phi(t_2)$  using the line passing through  $(t_1, y_1)$  with slope  $f(t_1, y_1)$ :

$$y_2 = y_1 + f(t_1, y_1)(t_2 - t_1)$$

- ✦ Thus we create a sequence  $y_n$  of approximations to  $\phi(t_n)$ :

$$y_1 = y_0 + f_0 \cdot (t_1 - t_0)$$

$$y_2 = y_1 + f_1 \cdot (t_2 - t_1)$$

⋮

$$y_{n+1} = y_n + f_n \cdot (t_{n+1} - t_n)$$

where  $f_n = f(t_n, y_n)$ .

- ✦ For a uniform step size  $h = t_n - t_{n-1}$ , Euler's formula becomes

$$y_{n+1} = y_n + f_n h, \quad n = 0, 1, 2, \dots$$

## Error Bounds and Numerical Methods

- ✦ In using a numerical procedure, keep in mind the question of whether the results are accurate enough to be useful.
- ✦ In our examples, we compared approximations with exact solutions. However, **numerical procedures are usually used when an exact solution is not available**. What is needed are bounds for (or estimates of) errors, which do not require knowledge of exact solution. More discussion on these issues and other numerical methods is given in Chapter 8.
- ✦ Since numerical approximations ideally reflect behavior of solution, a member of a diverging family of solutions is harder to approximate than a member of a converging family.
- ✦ Also, direction fields are often a relatively easy first step in understanding behavior of solutions.