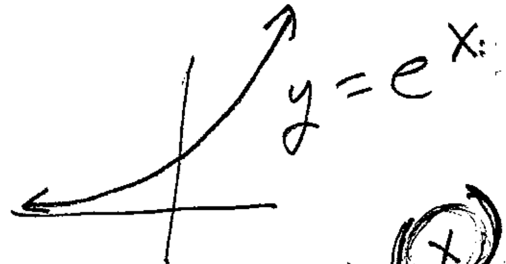
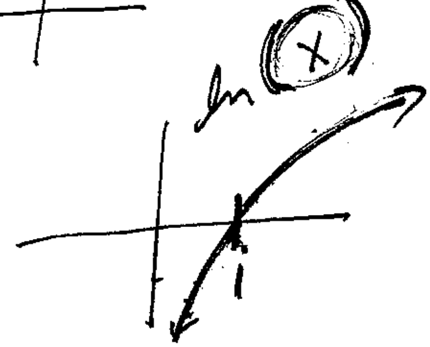
Double Quiz 11 (Show all work)

Find the following for $f(x) =$ (if they exist; if they don't exist, state so). Use this information to graph f.

Note $f'(x) =$ and $f''(x) =$

$+$ $-$ $+$ $-$





[2] 1a.) relative maximum(s) occur at $x =$ ____________________

[3] 1b.) The absolute maximum of f on the interval $[0, 5]$? ? is _______ and occurs at $x =$ ______________.

[3] 1c.) The absolute maximum of f is _______ and occurs at $x =$ ______________.

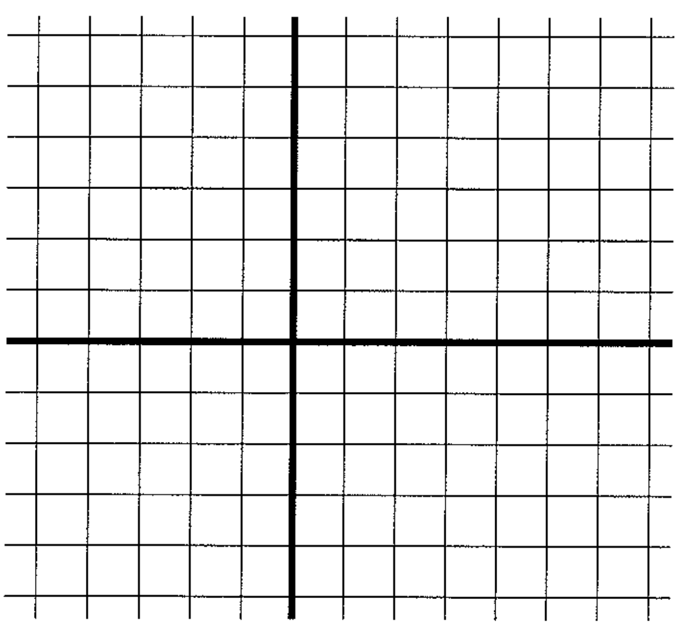
[2] 1d.) f is increasing on the intervals ____________________

[2] 1e.) f is concave up on the intervals ____________________

[3] 1f.) Equation(s) of vertical asymptote(s)____________________

[3] 1g.) Equation(s) of horizontal asymptote(s)____________________

[7] 1h.) Graph f



[10] 2.) related rates 3.7

[10] 3.) integration by substitution 5.5

2 ways to solve D.E

1) Ch 5

2) 8.4

2 ways to analyze DE

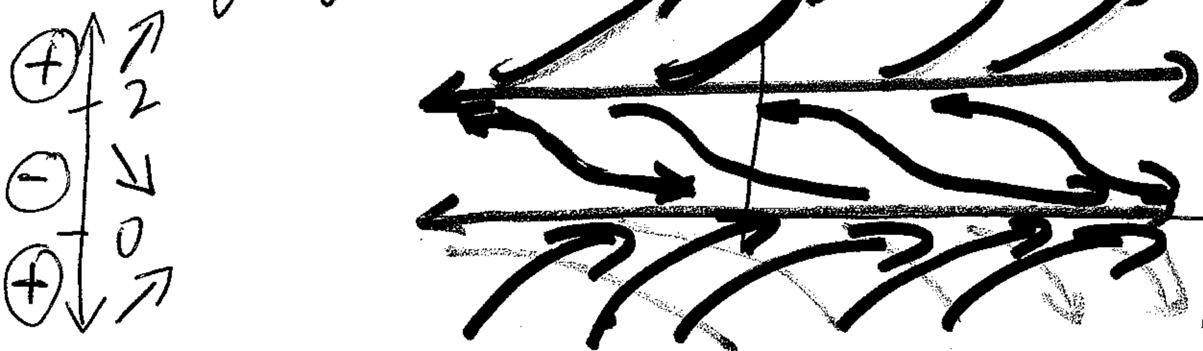
1) Direction field (8.1)

exam poly f(y) → 2) 8.3 $y' = f(y)$

EX: $y' = y(y-2)$

Equil. sol'n: $y = c \iff y' = 0$
↑ constant

$0 = y(y-2) \Rightarrow y = 0,\ y = 2$



unstable $y = 2$

$y = 0$ stable

2

CH 8: Solve $\boxed{\frac{dy}{dt} = f(t,y)}$

#4 ***8.1: Direction Fields ****************************** #8

Ch 5: Solve $\frac{dy}{dt} = f(t)$.

$y' = e^{2t}$

Suppose f is continuous

$$y = \frac{e^{2t}}{2} + C$$

$$dy = f(t)dt$$

$$\int dy = \int f(t)dt$$

$\boxed{y = F(t) + C}$ where F is any anti-derivative of F.

Initial Value Problem (IVP): $y(t_0) = y_0$

$$y_0 = F(t_0) + C \text{ implies } C = y_0 - F(t_0)$$

Hence unique solution (if domain connected) to IVP:

$$y = F(t) + y_0 - F(t_0)$$

Section 8.3: Solve $\frac{dy}{dt} = f(y)$

If given either differential equation $y' = f(y)$ OR direction field:

Find equilibrium solutions and determine if stable, unstable, semi-stable.

Understand what the above means.

Weird example

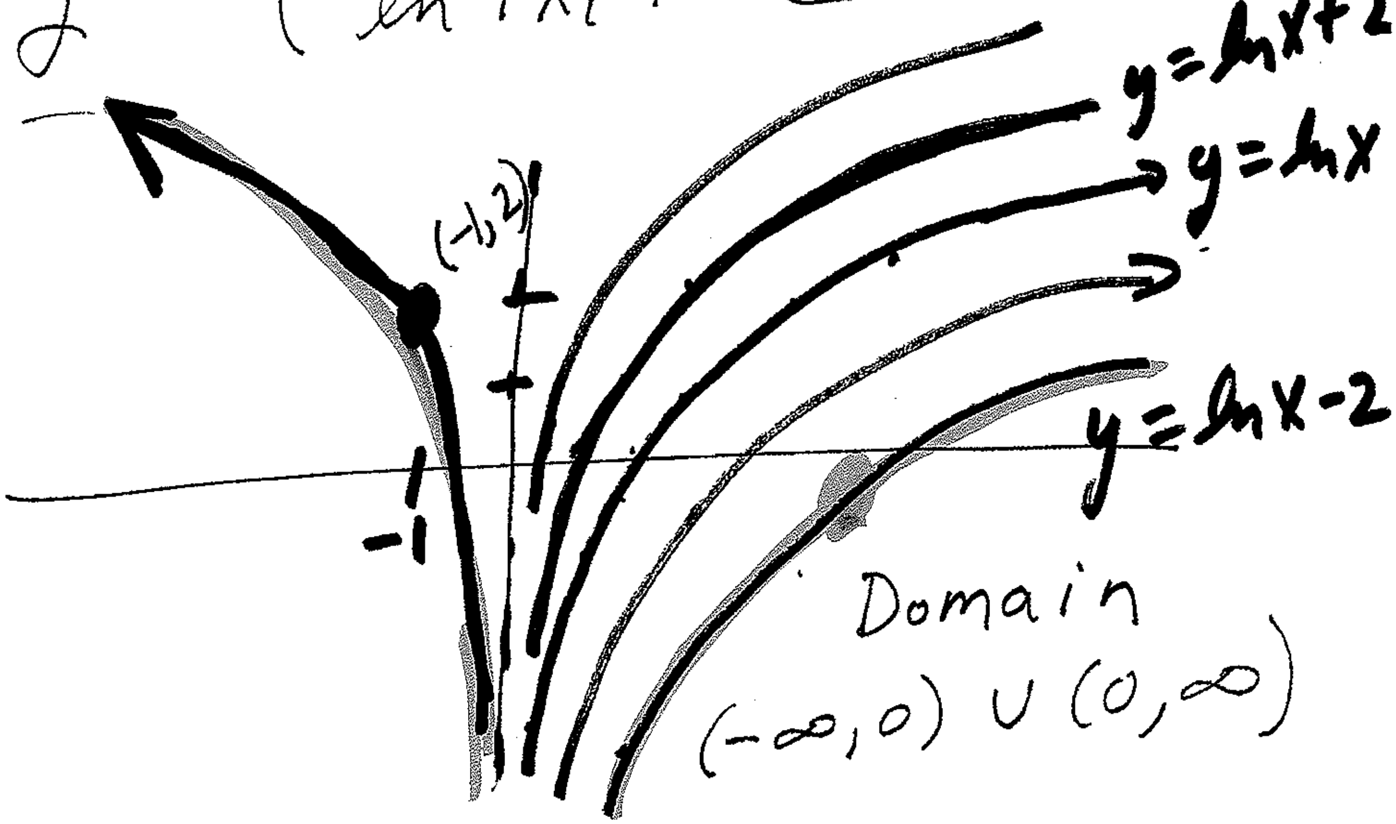
IVP: $y' = \frac{1}{x}$; $y(-1) = 2$

$$y = \ln|x| + C$$

$y(-1) = 2$: $\quad 2 = \cancelto{0}{\ln|-1|} + C$

$$\Rightarrow C = 2$$

$$y = \begin{cases} \ln|x| + 2 & \text{if } x < 0 \\ \ln|x| + C & \text{if } x > 0 \end{cases}$$



Domain

$(-\infty, 0) \cup (0, \infty)$

$y \quad y' \quad y''$

8.2 Linear vs Non-linear

nonlinear: $y' = y^2$

linear: $a_0(t)y^{(n)} + ... + a_n(t)y = g(t)$

Ex: $t\boxed{y''} - t^3\boxed{y'} - 3\boxed{y} = sin(t)$ linear

Ex: $2y'' - 3y' - 3y^2 = 0$ non-linear

First order linear eqn: $y' + p(t)y = g(t)$

Ex 1: $\boxed{y'} = a\boxed{y} + b$

Ex 2: $\boxed{y'} + 3t^2\boxed{y} = t^2$, $y(0) = 0$

Note: could use section 8.4 method, separation of variables to solve ex 1 and 2.

Ex 3: $t^2\boxed{y'} + 2t\boxed{y} = sin(t)$ can't use separation of variables

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$\boxed{\text{FYI}}$ Solve $t^2y' + 2ty = sin(t)$
(note, cannot use separation of variables).

$t^2y' + 2ty = sin(t)$

product rule

$(t^2y)' = sin(t)$

$\int (t^2y)'dt = \int sin(t)dt$

$(t^2y) = -cos(t) + C$ implies $y = -t^{-2}cos(t) + Ct^{-2}$

FTC.

$\boxed{\text{FYI:}}$ Solve $y' + p(x)y = g(x)$

Let $F(x)$ be an anti-derivative of $p(x)$

$e^{F(x)}y' + [p(x)e^{F(x)}]y = g(x)e^{F(x)}$

$e^{F(x)}y' + [F'(x)e^{F(x)}]y = g(x)e^{F(x)}$

$[e^{F(x)}y]' = g(x)e^{F(x)}$

$e^{F(x)}y = \int g(x)e^{F(x)}dx$

$y = e^{-F(x)} \int g(x)e^{F(x)}dx$

Two methods for analyt diff eqn w/out solving
1) 8.1: direction fields
2) Special case $y' = f(y)$
8.3 eg: sol'n

8.4: Solve $\frac{dy}{dt} = ay + b$ by separating variables: $\frac{dy}{ay+b} = dt$

$\int \frac{dy}{ay+b} = \int dt$

$\frac{ln|ay+b|}{a} = t + C$

$ln|ay + b| = at + C$

$e^{ln|ay+b|} = e^{at+C}$

$|ay + b| = e^C e^{at}$

$ay + b = \pm(e^C e^{at})$

$ay = Ce^{at} - b$ implies $y = Ce^{at} - \frac{b}{a}$

Two methods for solving diff eqns
1) Ch 5:
2) Separation of variables 8.4

Ex 2: $y' + 3t^2 y = t^2$, $y(0) = 0$

$$\frac{dy}{dt} + 3t^2 y = t^2$$

$$\frac{dy}{dt} = t^2 - 3t^2 y$$

$$dy = (t^2 - 3t^2 y)dt$$

$$dy = (1 - 3y)(t^2 dt)$$

$$\int \frac{dy}{1-3y} = \int t^2 dt$$

u-sub

$$-\frac{1}{3} ln|1 - 3y| = \frac{t^3}{3} + C$$

$$ln|1 - 3y| = -t^3 + C$$

$$|1 - 3y| = e^{-t^3 + C}$$

$$|1 - 3y| = e^C e^{-t^3}$$

$$1 - 3y = \pm e^C e^{-t^3}$$

$$1 - 3y = Ce^{-t^3} - 1$$

$$-3y = -Ce^{-t^3} - 1$$

$$\boxed{y = \frac{1 - Ce^{-t^3}}{3} = \frac{1}{3} - Ce^{-t^3}}$$

$$0 = \frac{1}{3} - Ce^0 \Rightarrow C = \frac{1}{3}$$

$$x^{ab} = (x^b)^a$$

$$\boxed{e^{-\frac{1}{3} ln|1-3y|} = e^{-\frac{t^3}{3} + C}}$$

$$(e^{ln|1-3y|})^{-\frac{1}{3}} = e^{-\frac{t^3}{3} + C}$$

$$e^{ln(|1 - 3y|^{-\frac{1}{3}})} = e^{\frac{t^3}{3} + C}$$

$$|1 - 3y|^{-\frac{1}{3}} = e^{-\frac{t^3}{3} + C}$$

$$|1 - 3y|^{-\frac{1}{3}} = e^C e^{\frac{t^3}{3}}$$

$$(|1 - 3y|^{-\frac{1}{3}})^{-3} = (e^C e^{\frac{t^3}{3}})^{-3}$$

$$|1 - 3y| = (e^{-3C} e^{-t^3})$$

$$1 - 3y = Ce^{-t^3}$$

$$y(0) = 0$$

$$\boxed{y = \frac{1}{3} - \frac{1}{3} e^{-t^3}}$$

*************$\boxed{\text{Existence of solution}}$*************

Ch 5) $y' = f(t)$, solution exists if f is continuous
$y = F(t) + C$ where F is any anti-derivative of f.

8.2) linear: $y' + p(x)y = g(x)$, solution exists if p and g are continuous

Ch 8): $y' = f(t, y)$, solution may or may not exist.

Ex: $\cancel{y'} = \cancel{y'} + 1 \Rightarrow 0 = 1 \Rightarrow$ no sol'n

Ex: $(y')^2 = -1 \Rightarrow$ no real sol'n

IVP ex: $\boxed{\frac{dy}{dx} = y(1 + \frac{1}{x}),\ y(0) = 1}$

$\int \frac{dy}{y} = \int (1 + \frac{1}{x})dx$

$ln|y| = x + ln|x| + C$

$|y| = e^{x+ln|x|+C} = e^x e^{ln|x|} e^C = C|x|e^x = Cxe^x$

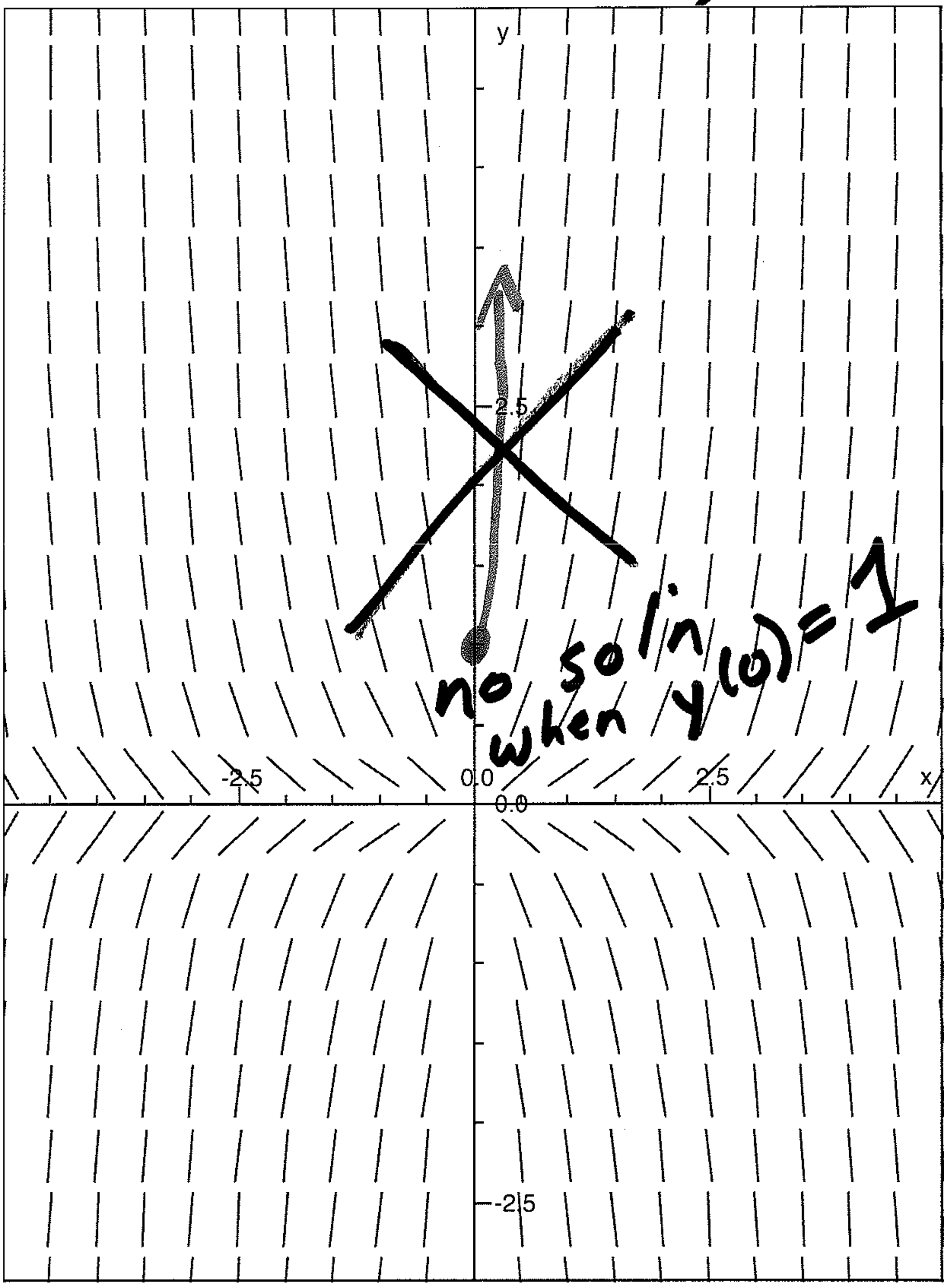
$y = \pm Cxe^x$ implies $y = Cxe^x$

$y(0) = 1: \quad 1 = C(0)e^0 = 0$ implies

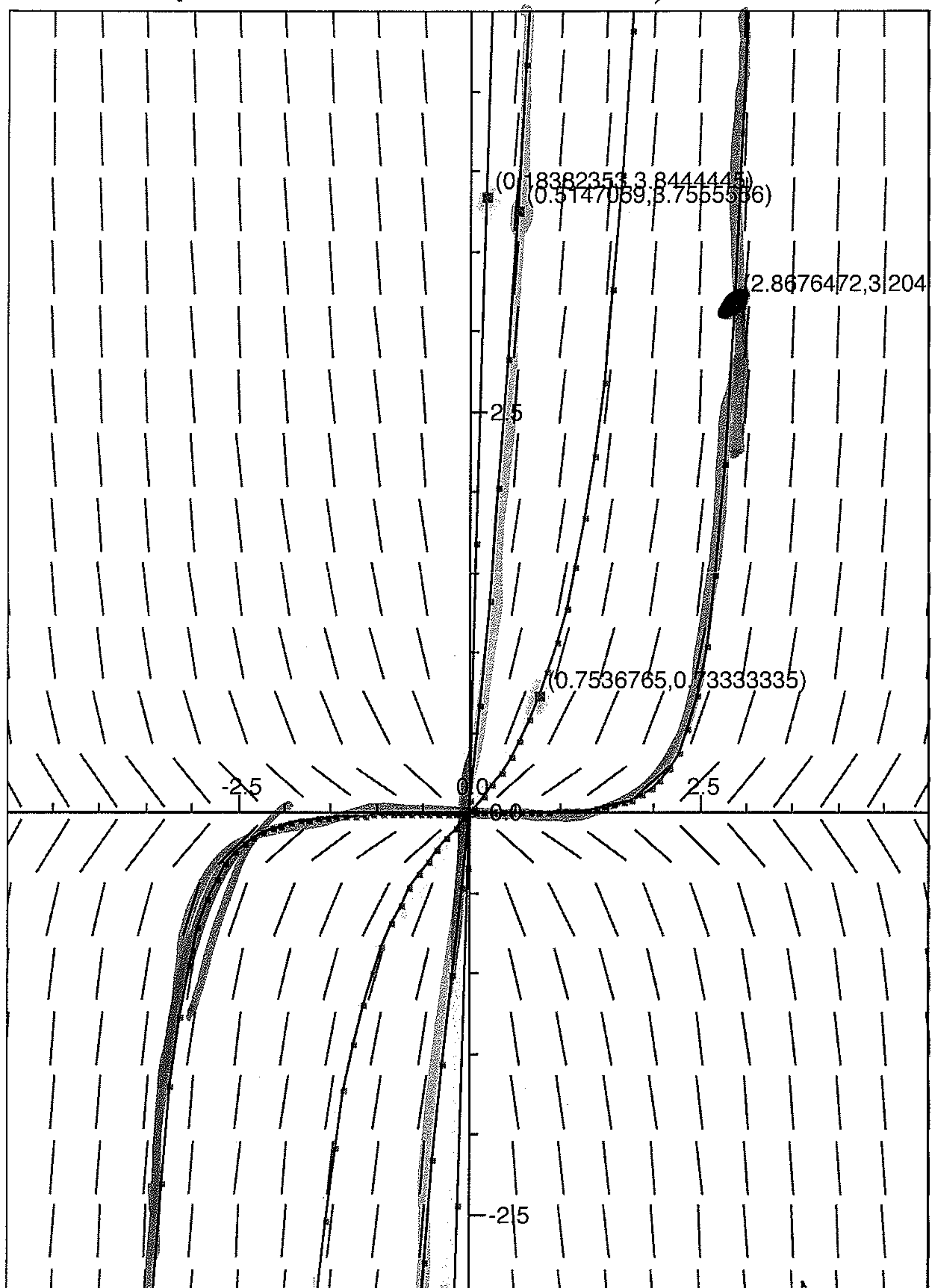
IVP $\frac{dy}{dx} = y(1 + \frac{1}{x})$, $y(0) = 1$ has $\boxed{\text{no solution.}}$

See direction field created using
www.math.rutgers.edu/~ sontag/JODE/JOdeApplet.html

$$y' = y\left(1+\frac{1}{x}\right)$$



$$y' = y\left(1 + \frac{1}{x}\right)$$



a IVP $y(0) = 1 \leftarrow$ no soln

IVP $y(0) = 0 \leftarrow$ infinite # of solns

**************$\boxed{\text{Uniqueness of solution}}$**************

Given an initial value problem,

Ch 5) $y' = f(t)$, $y(t_0) = y_0$: if f continuous, then on appropriate domain, unique solution $y = F(t) + y_0 - F(t_0)$. ✓

8.2) linear: $y' + p(x)y = g(x)$, then on appropriate domain, unique solution if p and g are continuous . ✓

Ch 8): $\boxed{y' = f(t, y), \text{ solution may or may not be unique.}}$

Ex: $y' = y^{\frac{1}{3}}$ EX: $y' = y(1 + \frac{1}{x})$, $y(0) = 0$

Note $\boxed{y = 0}$ is a solution to $y' = y^{\frac{1}{3}}$ since $y' = 0 = 0^{\frac{1}{3}} = y^{\frac{1}{3}}$

Suppose $y \neq 0$. Then $\frac{dy}{dx} = y^{\frac{1}{3}}$ implies $y^{-\frac{1}{3}} dy = dx$

$\int y^{-\frac{1}{3}} dy = \int dx$ implies $\frac{3}{2} y^{\frac{2}{3}} = x + C$

$y^{\frac{2}{3}} = \frac{2}{3}x + C$ implies $y = \pm\sqrt{(\frac{2}{3}x + C)^3}$

Suppose $y(0) = 4$. Then $4 = \sqrt{C^3}$ implies $C = 4^{\frac{2}{3}}$.

Thus initial value problem, $y' = y^{\frac{1}{3}}$, $y(0) = 4$, has 3 sol'ns:

$\boxed{y = 0}$, $\boxed{y = \sqrt{(\frac{2}{3}x + 4^{\frac{2}{3}})^3}}$, $\boxed{y = -\sqrt{(\frac{2}{3}x + 4^{\frac{2}{3}})^3}}$

$y' = y^{1/3}$

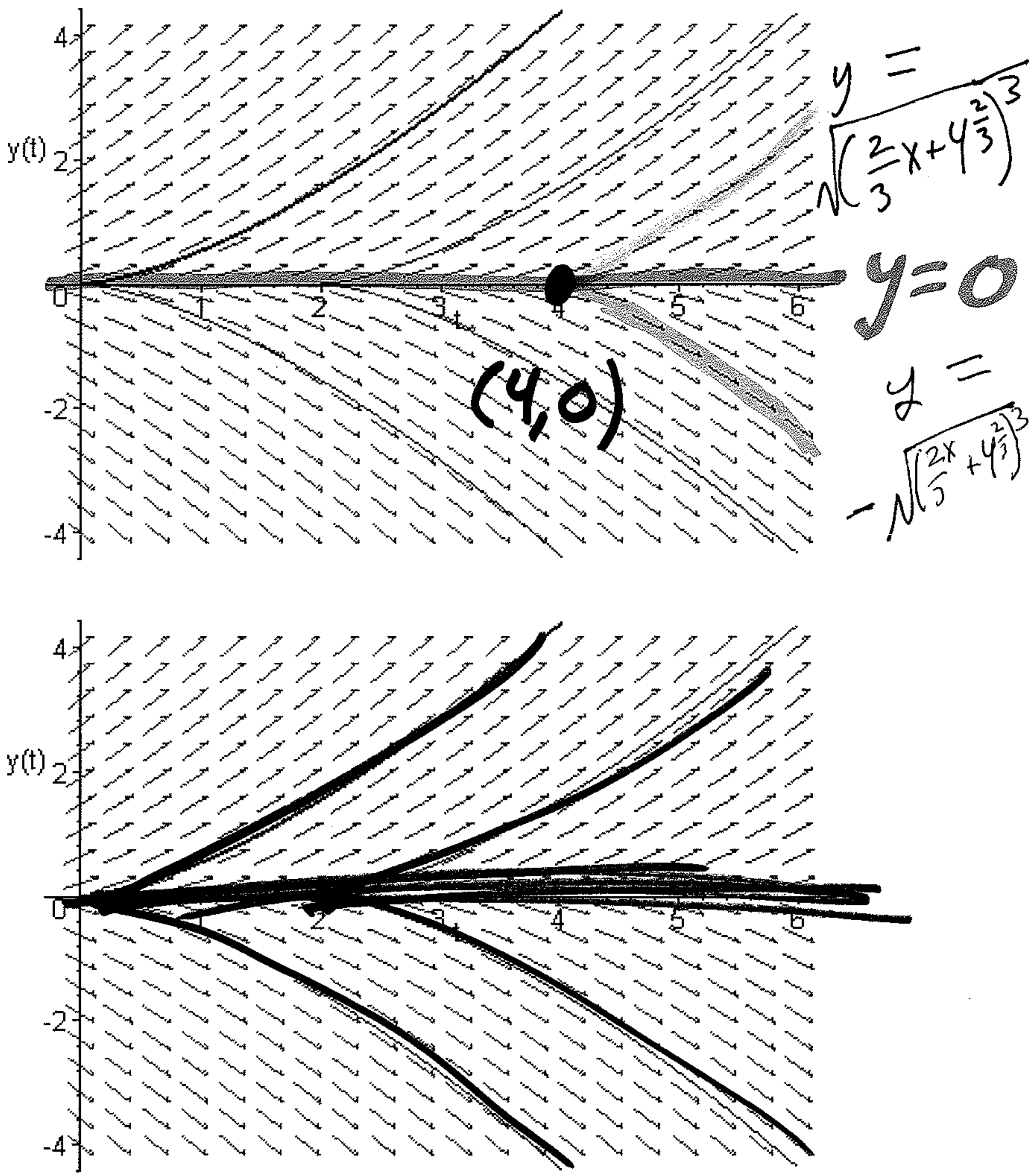


Figure 2.4.1 from ***Elementary Differential Equations and Boundary Value Problems*****, Eighth Edition** by William E. Boyce and Richard C. DiPrima

8.5

Boyce/DiPrima 9th ed, Ch 2.7: Numerical Approximations: Euler's Method

Elementary Differential Equations and Boundary Value Problems, 9th edition, by William E. Boyce and Richard C. DiPrima, ©2009 by John Wiley & Sons, Inc.

direction fields

- Recall that a first order initial value problem has the form

$$\frac{dy}{dt} = f(t, y), \ y(t_0) = y_0$$

- If f and $\partial f / \partial y$ are continuous, then this IVP has a unique solution $y = \phi(t)$ in some interval about t_0.
- When the differential equation is linear, separable or exact, we can find the solution by symbolic manipulations.
- However, the solutions for most differential equations of this form cannot be found by analytical means.
- Therefore it is important to be able to approach the problem in other ways.

Numerical Methods

use computer

- For our first order initial value problem

$$y' = f(t, y), \ y(t_0) = y_0,$$

an alternative is to compute approximate values of the solution $y = \phi(t)$ at a selected set of t-values.
- Ideally, the approximate solution values will be accompanied by error bounds that ensure the level of accuracy.
- There are many numerical methods that produce numerical approximations to solutions of differential equations, some of which are discussed in Chapter 8. 8.5
- In this section, we examine the **tangent line method**, which is also called **Euler's Method**.

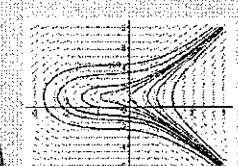
General Error Analysis Discussion (1 of 4)

- Recall that if f and $\partial f / \partial y$ are continuous, then our first order initial value problem

$$y' = f(t, y), \ y(t_0) = y_0$$

has a solution $y = \phi(t)$ in some interval about t_0.
- In fact, the equation has infinitely many solutions, each one indexed by a constant c determined by the initial condition.
- Thus ϕ is the member of an infinite family of solutions that satisfies $\phi(t_0) = y_0$.

Check to see if a solution exists;
Otherwise you might find a solution which doesn't actually exist.



Euler's Method: Tangent Line Approximation

- For the initial value problem

$$y' = f(t, y), \ y(t_0) = y_0,$$

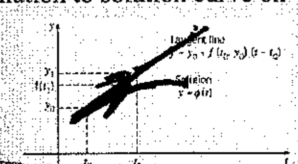
point on line

we begin by approximating solution $y = \phi(t)$ at initial point t_0.
- The solution passes through initial point (t_0, y_0) with slope $f(t_0, y_0)$. The line tangent to the solution at this initial point is

$$y = y_0 + f(t_0, y_0)(t - t_0)$$

- The tangent line is a good approximation to solution curve on an interval short enough.
- Thus if t_1 is close enough to t_0, we can approximate $\phi(t_1)$ by

$$y_1 = y_0 + f(t_0, y_0)(t_1 - t_0)$$



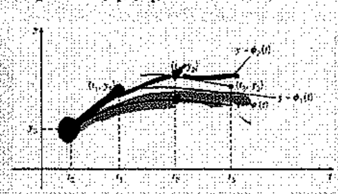
3.6: linearizt ion

slope

Approx fn w/ tangent line

General Error Analysis Discussion (2 of 4)

- The first step of Euler's method uses the tangent line to ϕ at the point (t_0, y_0) in order to estimate $\phi(t_1)$ with y_1.
- The point (t_1, y_1) is typically not on the graph of ϕ, because y_1 is an approximation of $\phi(t_1)$.
- Thus the next iteration of Euler's method does not use a tangent line approximation to ϕ, but rather to a nearby solution ϕ_1 that passes through the point (t_1, y_1).
- Thus Euler's method uses a succession of tangent lines to a sequence of different solutions $\phi, \phi_1, \phi_2, \ldots$ of the differential equation.



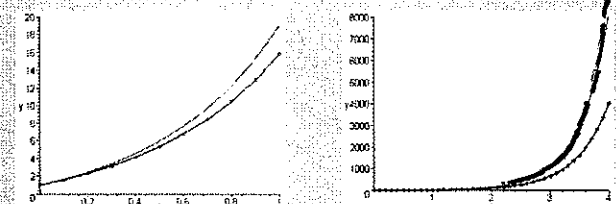
Example 3: Error Analysis & Graphs (3 of 3)

- Given below are graphs showing the exact solution (red) plotted together with the Euler approximation (blue).

t	Exact y	Approx y	Error	% Rel Error
0.00	1.00	1.00	0.00	0.00
1.00	19.07	15.78	3.29	17.27
2.00	149.39	104.68	44.72	29.93
3.00	1109.18	652.53	456.64	41.17
4.00	8197.88	4042.12	4155.76	50.69

Exact Solution:

$$y = -\frac{7}{4} + \frac{1}{2}t + \frac{11}{4}e^{2t}$$



Euler's Formula

- For a point t_2 close to t_1, we approximate $\phi(t_2)$ using the line passing through (t_1, y_1) with slope $f(t_1, y_1)$:

$$y_2 = y_1 + f(t_1, y_1)(t_2 - t_1)$$

- Thus we create a sequence y_n of approximations to $\phi(t_n)$:

$$y_1 = y_0 + f_0 \cdot (t_1 - t_0)$$
$$y_2 = y_1 + f_1 \cdot (t_2 - t_1)$$
$$\vdots$$
$$y_{n+1} = y_n + f_n \cdot (t_{n+1} - t_n)$$

where $f_n = f(t_n, y_n)$.

- For a uniform step size $h = t_n - t_{n-1}$, Euler's formula becomes

$$y_{n+1} = y_n + f_n h, \quad n = 0, 1, 2, \ldots$$

Error Bounds and Numerical Methods

- In using a numerical procedure, keep in mind the question of whether the results are accurate enough to be useful.
- In our examples, we compared approximations with exact solutions. However, numerical procedures are usually used when an exact solution is not available. What is needed are bounds for (or estimates of) errors, which do not require knowledge of exact solution. More discussion on these issues and other numerical methods is given in Chapter 8.
- Since numerical approximations ideally reflect behavior of solution, a member of a diverging family of solutions is harder to approximate than a member of a converging family.
- Also, direction fields are often a relatively easy first step in understanding behavior of solutions.

Does sol'n exist?

$$y' = 2 \implies y = 2x + C$$

constant slope field $\implies$ line