Properties of the definite integral

\[ \int_{a}^{b} f(x) \, dx = 0 \]

\[ \int_{a}^{b} f(x) \, dx = -\int_{b}^{a} f(x) \, dx \]

\[ \int_{a}^{b} k f(x) \, dx = k \int_{a}^{b} f(x) \, dx \]

\[ \int_{a}^{b} (f_1 + f_2)(x) \, dx = \int_{a}^{b} f_1(x) \, dx + \int_{a}^{b} f_2(x) \, dx \]

If \( f_1(x) \leq f_2(x) \), then \( \int_{a}^{b} f_1(x) \, dx \leq \int_{a}^{b} f_2(x) \, dx \)

If \( m \leq f(x) \leq M \) then \( m(b-a) \leq \int_{a}^{b} f(x) \, dx \leq M(b-a) \)

\[ e^{a(b-a)} \leq \text{area} \leq e^{b(b-a)} \]
\[ m (b-a) \leq \int_a^b f(x) \, dx \leq M (b-a) \]

\[ \text{1 circumscribed rectangle} \]

any upper bound on \( f(x) \) can be used to bound the area \( \int_a^b f(x) \, dx \)
\[
\int_a^b f(x) \, dx = \text{net area between } y = f(x) \text{ and } y = 0 \text{ from } x = a \text{ to } x = b
\]

\[
\lim_{n \to \infty} \sum_{i=1}^{m} f(x_i) \Delta x
\]
Find the area between the curve $y^2 = 2x - 2$ and $y = x - 5$.

Use vertical rectangles:

1.) Find points of intersection between the two curves.

$y^2 = 2x - 2$ and $y = x - 5$.

$(x - 5)^2 = 2x - 2$

$x^2 - 10x + 25 = 2x - 2$

$x^2 - 12x + 27 = 0$

$(x - 3)(x - 9) = 0$. Hence $x = 3, 9$.

2.) Determine which is larger.

Between 1 and 3: $\sqrt{2x - 2} > -\sqrt{2x - 2}$

Between 3 and 9: $\sqrt{2x - 2} > x - 5$

3.) Write as integral(s)

Note that between 1 and 3, the height of the rectangles is $\sqrt{2x - 2} - (-\sqrt{2x - 2})$ and the width is $dx$.

Note that between 3 and 9, the height of the rectangles is $\sqrt{2x - 2} - (x - 5)$ and the width is $dx$.

$\int_{1}^{3} [\sqrt{2x - 2} - (-\sqrt{2x - 2})]dx + \int_{3}^{9} [\sqrt{2x - 2} - (x - 5)]dx$
4.) Evaluate the integral
\[
\int_1^3 [2\sqrt{2x - 2}] \, dx + \int_3^9 [\sqrt{2x - 2} - (x - 5)] \, dx
\]
\[
= \int_1^3 [2\sqrt{2x - 2}] \, dx + \int_3^9 (\sqrt{2x - 2}) \, dx - \int_3^9 (x - 5) \, dx
\]
Let \( u = 2x - 2, \, du = 2 \, dx, \)
\( x = 1 : u = 2(1) - 2 = 0; \)
\( x = 3 : u = 2(3) - 2 = 4; \)
\( x = 9 : u = 2(9) - 2 = 16 \)
\[
= \int_0^4 u^{\frac{1}{2}} \, du + \int_4^{16} \frac{1}{2} u^{\frac{1}{2}} \, du + \int_3^9 (-x + 5) \, dx
\]
\[
= \frac{2}{3} u^{\frac{3}{2}} \bigg|_0^4 + \frac{1}{3} u^{\frac{3}{2}} \bigg|_4^{16} + \left( -\frac{1}{2} x^2 + 5x \right) \bigg|_3^9
\]
\[
= \frac{2}{3} (4^{\frac{3}{2}} - 0^{\frac{3}{2}}) + \frac{1}{3} (16^{\frac{3}{2}} - 4^{\frac{3}{2}}) + \left( -\frac{1}{2} (9)^2 + 5(9) \right) - \left( -\frac{1}{2} (3)^2 + 5(3) \right)
\]
\[
= \frac{1}{3} [2(8) + 64 - 8] - \frac{81}{2} + 45 + \frac{9}{2} - 15 = 16
\]
\[
= \frac{72}{3} - \frac{72}{2} + 30 = 24 - 36 + 30 = 18
\]
Ex: Find area bounded between $y = -x^2 + 5$ and $y = 0$ (x-axis)

1. Find intersection points (⇒ same y-value)
   
   $0 = -x^2 + 5$
   
   $x^2 = 5 \Rightarrow x = \pm \sqrt{5}$

\[ \int_{-\sqrt{5}}^{\sqrt{5}} (\text{heights})(\text{widths}) \, dx \]
\[
\sqrt{5} \int_{-\sqrt{5}}^{\sqrt{5}} (-x^2 + 5) \, dx
\]

\[
= -\frac{x^3}{3} + 5x \bigg|_{-\sqrt{5}}^{\sqrt{5}}
\]

\[
= \left( -\frac{5\sqrt{5}}{3} + 5\sqrt{5} \right) - \left( \frac{5\sqrt{5}}{3} - 5\sqrt{5} \right)
\]

\[
= -10\sqrt{5} + 10\sqrt{5}
\]

\[
= \frac{20\sqrt{5}}{3}
\]
Find area bounded by
\[ y = -x^2 + 5 \quad \text{and} \quad y = 2x - 10 \]

0. Find intercepts (\( \Rightarrow \) same \( y \)-value)

\[ -x^2 + 5 = 2x - 10 \]

0 = \( x^2 + 2x - 15 \)

0 = (\( x + 5 \))(\( x - 3 \))

\( x = -5, \ 3 \)

\[ \int_{-5}^{3} (\text{heights}) \, dx \]
2. Determine which function is larger on \([-5, 3]\).

**Method 1**

Algebraically:

\[-x^2 + 5 > 2x - 10\]

Check:

\[x = 0\]

Since:

\[5 > -10\] when \[x = 6\]

\[-5 \quad 0 \quad 3\]

**Method 2**

Graphically:

\[y = -x^2 + 5\]

\[2x - 10\]

\[(x^2 - 5) - (2x - 10)\]
\[-x^2 + 5 > 2x - 10\]

\[\Rightarrow \text{height} = (-x^2 + 5) - (2x - 10)\]

\[\text{Area} \int_{-5}^{3} \left[(-x^2 + 5) - (2x - 10)\right] \, dx\]
Find the area bounded by the functions $y = 2x^3$ and $y = 2x^{\frac{1}{3}}$.

1.) Find points of intersection:

$2x^3 = 2x^{\frac{1}{3}}$ implies $x^3 = x^{\frac{1}{3}}$ implies $x^9 = x$. Thus $x^9 - x = x(x^8 - 1) = 0$.

Hence $x = 0$ and $x^8 - 1 = 0$. $x^8 = 1$ implies $x = 1, -1$

Hence the functions $y = 2x^3$ and $y = 2x^{\frac{1}{3}}$ intersect when $x = -1, 0, 1$

2.) Draw a rough graph:

3.) Find area:

Use vertical rectangles:

$\int_{-1}^{0} [2x^3 - 2x^{\frac{1}{3}}]dx + \int_{0}^{1} [2x^{\frac{1}{3}} - 2x^3]dx$
Ex: Find area btwn 
\[ y = 2x^3 \quad \text{and} \quad y = 2x^{1/3} \]

(0) Find intercepts

(\Rightarrow \text{same } y\text{-value})

\[ 2x^3 = 2x^{1/3} \]
\[ x^3 = x^{1/3} \]
\[ x^9 = x \]
\[ x^9 - x = 0 \]
\[ x(x^8 - 1) = 0 \]
\[ x = 0, \quad x^8 - 1 = 0 \]
\[ x = 0, \quad x^8 = 1 \]
\[ x = 0, \quad x = \pm 1 \]

check

\[ 2(\pm 1)^3 = 2(\pm 1)^{1/3} \]
\[ 2(0)^3 = 2(0)^{1/3} \]
Alternate

\[ x^3 = x^{1/3} \]
\[ x^3 - x^{1/3} = 0 \]
\[ x^{1/3} (x^{8/3} - 1) = 0 \]
\[ x^{8/3} = 1 \implies x = \pm 1 \]

\[ x = 0 \]

2) Determine height
Find which fn is larger
Which is larger on $[-1, 0]$?

Algebraically:

$$2x^3 \geq 2x^{1/3}$$

$$2\left(-\frac{1}{8}\right)^3 > 2\left(-\frac{1}{8}\right)^{1/3} = -1$$

Which fn is larger on $[0, 1]$?

$$2\left(\frac{1}{8}\right)^3 < 2\left(\frac{1}{8}\right)^{1/3} = 1$$

$$2x^3 < 2x^{1/3}$$

$$\int_{-1}^{0} [2x^3 - 2x^{1/3}] \, dx + \int_{0}^{1} [2x^{1/3} - 2x^3] \, dx = \text{Area}$$