$a^x a^y = a^{x+y}$ $\qquad$ $\frac{a^x}{a^y} = a^{x-y}$

$(a^x)^y = a^{xy}$ $\qquad$ $(ab)^x = a^x b^x$

Ex: $2^8 5^6 =$

Suppose $f(x) = a^x$
$y = a^x$

Find f^{-1}

Switch x and y: $a^y = x$

$log_a x = y$ iff $a^y = x$

$f^{-1}(f(x)) =$

$f(f^{-1}(x)) =$

$$\underbrace{\begin{aligned}(e^x)' &= e^x \\ (e^{3x})' &= \left[(e^x)^3\right]' \\ & 3(e^x)^2 \cdot e^x \\ &= 3e^{2x} \cdot e^x \\ &= 3e^{3x}\end{aligned}}_{\text{long method}}$$

or equivalently

$$(e^{3x})' = e^{3x} \cdot (3)$$

$log_a x + log_a y = log_a(xy)$

$log_a x - log_a y = log_a(\frac{x}{y})$

$log_a x^r = r log_a(x)$

$log_a a =$ $\qquad$ $log_a 1 =$ $\qquad$ $log_a 0 =$

Defn: $ln(x) = log_e x$

$log_a x = \frac{ln(x)}{ln(a)}$

Note: $log_a x + log_a y \neq log_a(x+y)$

3.7

By the chain rule $[(x^2+1)^9]' = 9(x^2+1)^8(2x)$

Or in other words, $\frac{d[(x^2+1)^9]}{dx} = 9(x^2+1)^8(2x)$

Or in other words, if we let $u = x^2+1$, then

$$\frac{du}{dx} = u' = 2x \text{ and}$$

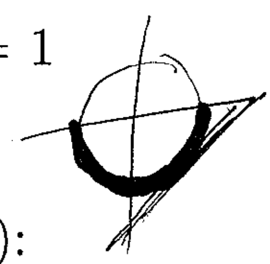
$$[(x+1)^9]' = [u^9]' = 9u^8u' = 9(x^2+1)^8(2x)$$

Or in other notation,

$$\frac{d[(x+1)^9]}{dx} = \frac{d[u^9]}{dx} = 9u^8\frac{du}{dx} = 9(x^2+1)^8(2x)$$

We can use the chain rule to calculate a derivative using implicit differentiation.

Ex: Find the slope of the tangent line to $x^2+y^2=1$ at the point $(x,y) = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$.



Long method (without using implicit differentiation):

Solve for y:

$x^2+y^2=1$ implies $y^2 = 1-x^2$ implies $y = \pm\sqrt{1-x^2}$

Since the $y-$value of the point $(x,y) = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ is negative, we are interested in the bottom half of the circle:

$$y = -\sqrt{1-x^2} = -(1-x^2)^{\frac{1}{2}}$$

To find slope of the tangent line, take derivative:

$$\frac{dy}{dx} = -\frac{1}{2}(1-x^2)^{\frac{-1}{2}}(-2x) = \frac{x}{\sqrt{1-x^2}}$$

Hence when $x = \frac{1}{\sqrt{2}}$, then the slope of the tangent line is

$$\frac{\frac{1}{\sqrt{2}}}{\sqrt{1-(\frac{1}{\sqrt{2}})^2}} = \frac{\frac{1}{\sqrt{2}}}{\sqrt{1-(\frac{1}{2})}} = \frac{\frac{1}{\sqrt{2}}}{\sqrt{\frac{1}{2}}} = \frac{\sqrt{\frac{1}{2}}}{\sqrt{\frac{1}{2}}} = 1$$

We can instead use implicit differentiation:

Note that y is a function of x for the bottom half of the circle: $y = f(x) = -(1-x^2)^{\frac{1}{2}}$

wrt x

Thus to find the derivative of y^2, we can use the chain rule:

$$\frac{d(y^2)}{dx} = 2y \cdot \frac{dy}{dx} = 2(-(1-x^2)^{\frac{1}{2}}) \cdot \frac{x}{\sqrt{1-x^2}} = -2x$$

Note $y^2 = [-(1-x^2)^{\frac{1}{2}}]^2 = 1-x^2$

However, we don't need the above to find the slope of the tangent line to the unit circle at the point $(x, y) = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$. Instead:

notation

Shorter method for finding this slope of the tangent line to $x^2 + y^2 = 1$ at the point $(x, y) = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$.

We have $x^2 + y^2 = 1$, and we want to find slope $= \frac{dy}{dx}$

Take the derivative with respect to x of both sides:

$$\frac{d(x^2 + y^2)}{dx} = \frac{d(1)}{x}$$

SIDE NOTE

$$\frac{d}{dx}\left(x^2\right) = 2x \cdot \frac{dx}{dx}$$

$$= 2x$$

$$2x + 2y \cdot \frac{dy}{dx} = 0$$

Solve for $\frac{dy}{dx}$: $\qquad 2y \cdot \frac{dy}{dx} = -2x$

$$\frac{dy}{dx} = \frac{-x}{y}$$

Hence the slope of the tangent line to the unit circle at the point $(x, y) = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$.

$$\frac{dy}{dx} = \frac{-x}{y} = \frac{-\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}} = 1$$

Suppose $2x^2y - 3y^2 = 4$. First find y':

Easiest method is to use implicit differentiation. Take derivative (with respect to x) of both sides.

$$\frac{d}{dx}(2x^2y - 3y^2) = \frac{d}{dx}(4)$$

$$4xy + 2x^2y' - 6yy' = 0$$

Solve for y' (note this step is easy as one can factor y' from some terms. Observe that this will always be the case):

$$y'(2x^2 - 6y) = -4xy$$

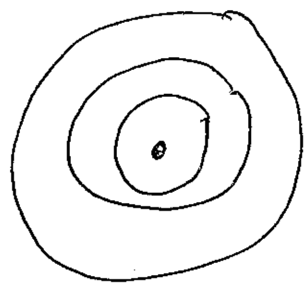
$$y' = \frac{-4xy}{2x^2-6y} = \frac{-2(2xy)}{-2(3y-x^2)} = \frac{2xy}{3y-x^2}$$

Hence $y' = \frac{2xy}{3y-x^2}$

3.7 : Related Rates

1.) A pebble dropped into a pond makes a circular wave that travels outward at a rate 0.4 meters per second. At what rate is the area of the circle increasing 2 seconds after the pebble strikes the pond?

0) Picture



$0.4 \, m/s = \frac{dr}{dt}$

1) What is the problem?
What do we need to find

$\frac{dA}{dt} = ?$ when $t = 2 sec$

2) Need an eqn involving A

$A = \pi r^2$

3^{0}) Simplify (sometimes)
often don't need to

3') Take the derivative
wrt $\underline{t}$ since need $\frac{dA}{dt}$

$$A = \pi r^2$$

$$\frac{dA}{dt} = \pi 2r \cdot \frac{dr}{dt}$$

4) Plug in values **for when $t=2$**

$$\frac{dr}{dt} = 0.4 \text{ m/sec}$$

$$r = \left(0.4 \frac{m}{\cancel{sec}}\right)\left(2 \cancel{sec}\right) = 0.8m$$

since dr/dt = constant

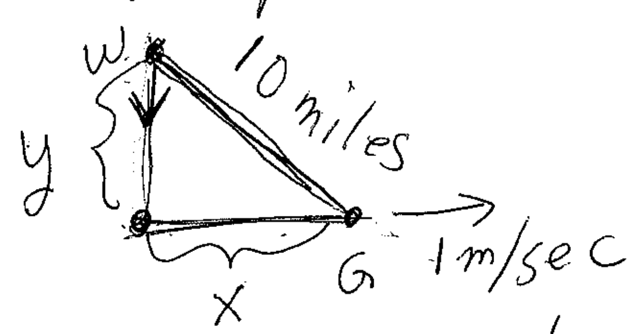
$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

$$= 2\pi (0.8\ m)\left(\frac{0.4\ m}{sec}\right)$$

$$= \boxed{0.64\pi\ m^2/sec}$$

2.) Suppose the distance between two planes must be maintained at 10 miles. Suppose plane W is north of a radio tower and moving south while plane G is east of the same radio tower. If plane G is moving east at 1 mile/second, how fast should plane W be moving when plane G is 6 miles from the radio tower? x'

0) Draw the picture



y = Distance W is from radio tower

X = Distance G is from radio tower

1) What do we need to find

$$\frac{dy}{dt} = ? \quad \text{when } x = 6$$

2) Equation involving y

$$\boxed{y^2 + x^2 = 10^2}$$

3,4) Take Derrivative wrt. t

$$\cancel{2}y\frac{dy}{dt} + \cancel{2}x\cdot\frac{dx}{dt} = 0$$

$$yy' + xx' = 0$$

Need y' when $x = 6$

Given $\frac{dx}{dt} = x' = 1$ mile/sec

$$y^2 + x^2 = 10^2$$

when $x = 6$: $y^2 + 36 = 100$

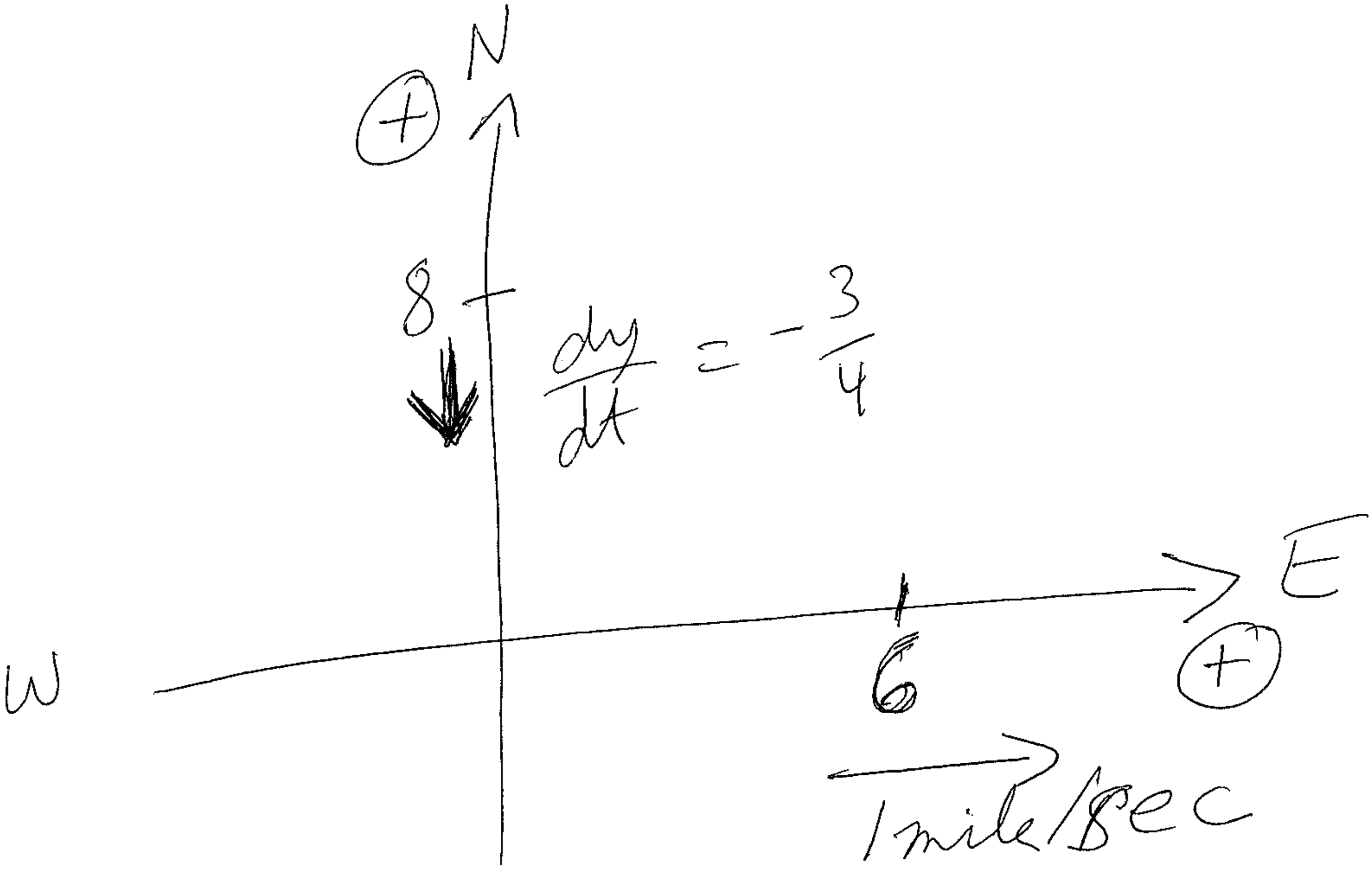
$$y^2 = 100 - 36 = 64$$

$$y = 8$$

$$yy' + xx' = 0$$

$$8y' + 6(1) = 0$$

$$y' = \frac{-6}{8} = \frac{-3}{4}$$



$$\frac{dy}{dt} = -\frac{3}{4}$$

1 mile/sec

$$\frac{dx}{dt} = +1$$

since moving east.

Plane W is moving south at rate of $\frac{3}{4}$ miles/sec

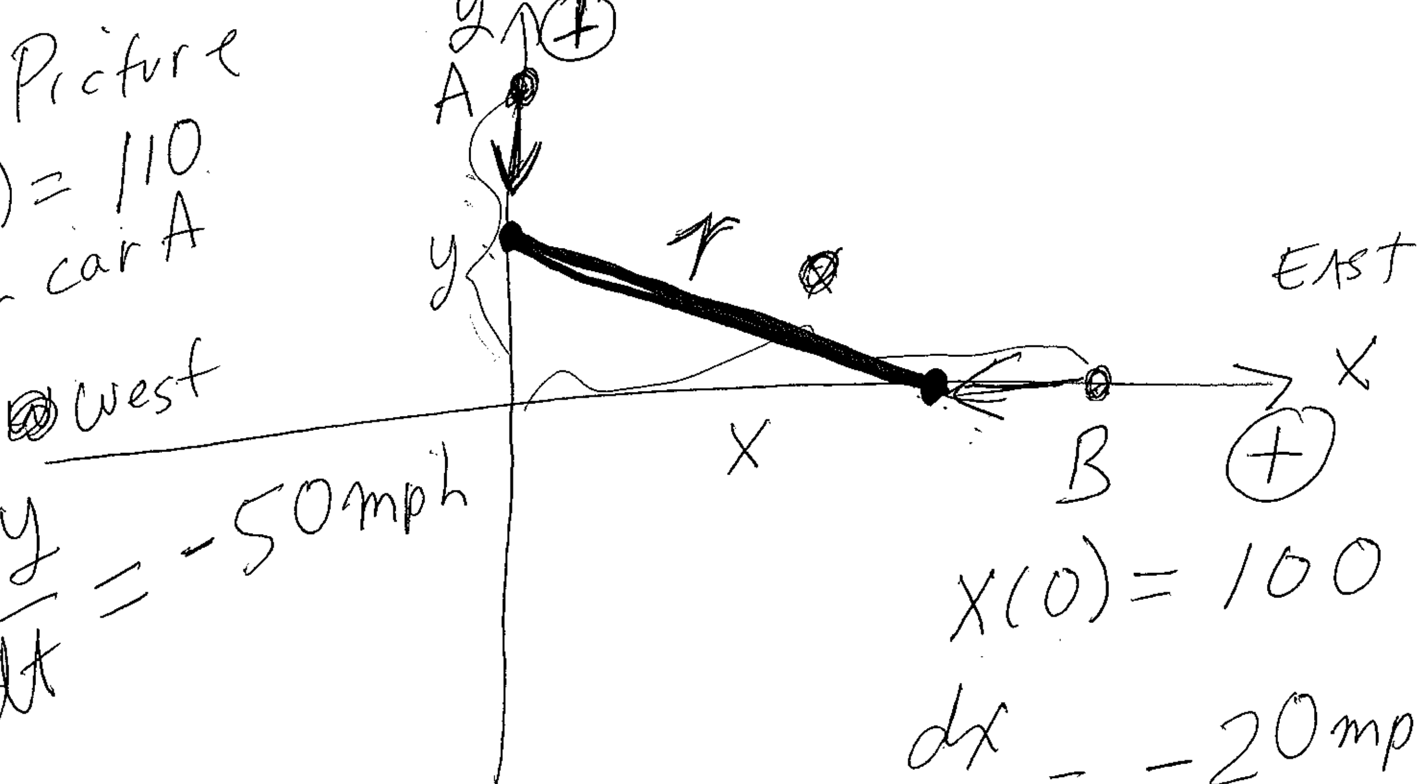
If moving west

$$\frac{dx}{dt} < 0$$

2.) Suppose car A is 110 miles north of an intersection and traveling south at 50 mph. Suppose car B is 100 miles east of the same intersection and traveling west at 20 mph. 1a.) At what rate are the cars approaching each other after 1 hour? 1b.) After 3 hours?

0) Picture

$y(0) = 110$ for car A



$\frac{dy}{dt} = -50$ mph

$x(0) = 100$

$\frac{dx}{dt} = -20$ mph

1) Find $\frac{dr}{dt}$ when a) $t = 1$

b) $t = 3$

2) Eqn involving r

$$x^2 + y^2 = r^2$$

Derivative: $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2r\frac{dr}{dt}$

$$\frac{dx}{dt} = -20 \qquad \frac{dy}{dt} = -50$$

a) $t = 1$

$x = \underline{80}$ $\quad y = \underline{60}$ $\quad r = \sqrt{}$