\[ a^x a^y = a^{x+y} \]

\[ \frac{a^x}{a^y} = a^{x-y} \]

\[(a^x)^y = a^{xy} \]

\[(ab)^x = a^x b^x \]

Ex: \(2^8 5^6 = \)

Suppose \(f(x) = a^x\)

\[ y = a^x \]

Find \(f^{-1}\)

Switch \(x\) and \(y\): \(a^y = x\)

\[ \log_a x = y \text{ iff } a^y = x \]

\[ f^{-1}(f(x)) = \]

\[ f(f^{-1}(x)) = \]

\[ \log_a x + \log_a y = \log_a(xy) \]

\[ \log_a x - \log_a y = \log_a \left(\frac{x}{y}\right) \]

\[ \log_a x^r = r \log_a (x) \]

\[ \log_a a = 1, \quad \log_a 1 = 0, \quad \log_a 0 = \text{undefined} \]

Defn: \(\ln(x) = \log_e x\)

\[ \log_a x = \frac{\ln(x)}{\ln(a)} \]

Note: \(\log_a x + \log_a y \neq \log_a (x + y)\)
3.7

By the chain rule \([(x^2 + 1)^9]' = 9(x^2 + 1)^8(2x)\)

Or in other words, \(\frac{d[(x^2 + 1)^9]}{dx} = 9(x^2 + 1)^8(2x)\)

Or in other words, if we let \(u = x^2 + 1\), then

\[\frac{du}{dx} = u' = 2x\] and

\[[(x + 1)^9]' = [u^9]' = 9u^8u' = 9(x^2 + 1)^8(2x)\]

Or in other notation,

\[\frac{d[(x+1)^9]}{dx} = \frac{d[u^9]}{dx} = 9u^8 \frac{du}{dx} = 9(x^2 + 1)^8(2x)\]

We can use the chain rule to calculate a derivative using implicit differentiation.

**Ex:** Find the slope of the tangent line to \(x^2 + y^2 = 1\) at the point \((x, y) = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})\).

**Long method (without using implicit differentiation):**

Solve for \(y\):

\(x^2 + y^2 = 1\) implies \(y^2 = 1 - x^2\) implies \(y = \pm \sqrt{1 - x^2}\)

Since the \(y\)-value of the point \((x, y) = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})\) is negative, we are interested in the bottom half of the circle:

\[y = -\sqrt{1 - x^2} = -(1 - x^2)^{\frac{1}{2}}\]
To find slope of the tangent line, take derivative:

\[
\frac{dy}{dx} = -\frac{1}{2} (1 - x^2)^{-\frac{1}{2}} (-2x) = \frac{x}{\sqrt{1-x^2}}
\]

Hence when \( x = \frac{1}{\sqrt{2}} \), then the slope of the tangent line is

\[
\frac{\frac{1}{\sqrt{2}}}{\sqrt{1-(\frac{1}{\sqrt{2}})^2}} = \frac{\frac{1}{\sqrt{2}}}{\sqrt{\frac{1}{2}}} = \frac{1}{\sqrt{\frac{1}{2}}} = \sqrt{2} = 1
\]

We can instead use implicit differentiation:

Note that \( y \) is a function of \( x \) for the bottom half of the circle: \( y = f(x) = -(1 - x^2)^{\frac{1}{2}} \)

Thus to find the derivative of \( y^2 \) with respect to \( x \), we can use the chain rule:

\[
\frac{d(y^2)}{dx} = 2y \cdot \frac{dy}{dx} = 2(- (1 - x^2)^{\frac{1}{2}}) \cdot \frac{x}{\sqrt{1-x^2}} = -2x
\]

Note \( y^2 = \left[-(1 - x^2)^{\frac{1}{2}}\right]^2 = 1 - x^2 \)

However, we don’t need the above to find the slope of the tangent line to the unit circle at the point \((x, y) = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})\). Instead:
**Shorter method** for finding this slope of the tangent line to \( x^2 + y^2 = 1 \) at the point \((x, y) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)\).

We have \( x^2 + y^2 = 1 \), and we want to find slope \( \frac{dy}{dx} \).

Take the derivative with respect to \( x \) of both sides:

\[
\frac{d(x^2 + y^2)}{dx} = \frac{d(1)}{x}
\]

\[
2x + 2y \cdot \frac{dy}{dx} = 0
\]

Solve for \( \frac{dy}{dx} \):

\[
2y \cdot \frac{dy}{dx} = -2x
\]

\[
\frac{dy}{dx} = \frac{-x}{y}
\]

Hence the slope of the tangent line to the unit circle at the point \((x, y) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)\).

\[
\frac{dy}{dx} = \frac{-x}{y} = \frac{-\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}} = 1
\]
Suppose $2x^2y - 3y^2 = 4$. First find $y'$:

Easiest method is to use implicit differentiation. Take derivative (with respect to $x$) of both sides.

$$\frac{d}{dx}(2x^2y - 3y^2) = \frac{d}{dx}(4)$$

$$4xy + 2x^2y' - 6yy' = 0$$

Solve for $y'$ (note this step is easy as one can factor $y'$ from some terms. Observe that this will always be the case):

$$y'(2x^2 - 6y) = -4xy$$

$$y' = \frac{-4xy}{2x^2-6y} = \frac{-2(2xy)}{-2(3y-x^2)} = \frac{2xy}{3y-x^2}$$

Hence $y' = \frac{2xy}{3y-x^2}$
3.7: Related Rates

1.) A pebble dropped into a pond makes a circular wave that travels outward at a rate 0.4 meters per second. At what rate is the area of the circle increasing 2 seconds after the pebble strikes the pond?

0) Picture

\[
0.4 \text{ m/s} = \frac{dr}{dt}
\]

1) What is the problem?
What do we need to find?

\[
\frac{dA}{dt} = ? \quad \text{when } t = 2 \text{ sec}
\]

2) Need an eqn involving A

\[
A = \pi r^2
\]
3) Simplify (sometimes) often don't need to

3') Take the derivative wrt \( t \) since need \( \frac{dA}{dt} \)

\[ A = \pi r^2 \]

\[ \frac{dA}{dt} = \pi \cdot 2r \cdot \frac{dr}{dt} \]

4) Plug in values for when \( t = 2 \)

\[ \frac{dr}{dt} = 0.4 \text{ m/sec} \]

\[ r = (0.4 \text{ m/sec}) \cdot (2 \text{ sec}) = 0.8 \text{ m} \]

Since \( \frac{dr}{dt} = \text{constant} \)
\[
\frac{dA}{dt} = 2\pi r \frac{dr}{dt}
\]

\[
= 2\pi (0.8 \text{ m}) (0.4 \text{ m/ sec})
\]

\[
= 0.64 \pi \text{ m}^2/\text{sec}
\]
2. Suppose the distance between two planes must be maintained at 10 miles. Suppose plane \( W \) is north of a radio tower and moving south while plane \( G \) is east of the same radio tower. If plane \( G \) is moving east at 1 mile/second, how fast should plane \( W \) be moving when plane \( G \) is 6 miles from the radio tower?

0) Draw the picture

\[ y = \text{Distance W is from radio tower} \]
\[ x = \text{Distance G is from radio tower} \]

1) What do we need to find

\[ \frac{dy}{dt} = ? \quad \text{when} \quad x = 6 \]

2) Equation involving \( y \)

\[ y^2 + x^2 = 10^2 \]

3, 4) Take Derivative \( \frac{dy}{dt} \)
\[ 2y \frac{dy}{dt} + 2x \frac{dx}{dt} = 0 \]

\[ y' + xx' = 0 \]

Need \( y' \) when \( x = 6 \)

Given \( \frac{dx}{dt} = x' = 1 \text{ mile/sec} \)

\[ y^2 + x^2 = 10^2 \]

when \( x = 6 \): \( y^2 + 36 = 100 \)

\[ y^2 = 100 - 36 = 64 \]

\[ y = 8 \]

\[ y' + xx' = 0 \]

\[ 8y' + 6(1) = 0 \]

\[ y' = -\frac{6}{8} = -\frac{3}{4} \]
Plane W is moving south at a rate of $\frac{3}{4}$ miles/sec. If moving west, $\frac{dx}{dt} < 0$. Since moving east, $\frac{dx}{dt} = +1$. $$\frac{dy}{dt} = -\frac{3}{4}$$
2.) Suppose car A is 110 miles north of an intersection and traveling south at 50 mph. Suppose car B is 100 miles east of the same intersection and traveling west at 20 mph. 1a.) At what rate are the cars approaching each other after 1 hour? 1b.) After 3 hours?

D) Find \( \frac{dr}{dt} \) when

a) \( t = 1 \)

b) \( t = 3 \)
2) Equation involving \( r \)

\[ x^2 + y^2 = r^2 \]

Derivative: 

\[ 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2r \frac{dr}{dt} \]

\[ \frac{dx}{dt} = -20 \quad \frac{dy}{dt} = -50 \]

\( a) \ t = 1 \)

\[ x = 80 \quad y = 60 \quad r = \sqrt{\_} \]

\( 110 \)

\( 50 \)