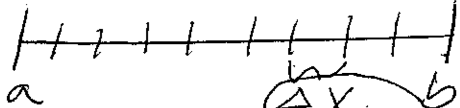


Suppose f integrable
(Note f continuous implies f integrable). $\Delta x = \frac{b-a}{n}$

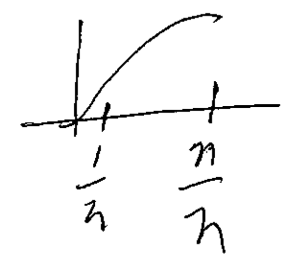
If n equal subdivisions: $\Delta x = \frac{b-a}{n}$ and if we use right-hand endpoints: $x_i = a + i\Delta x = a + \frac{(b-a)i}{n}$

$$\int_a^b f(x)dx = \underline{lim}_{n\to\infty}\left(\Sigma_{i=1}^{n} \underbrace{f(a+\tfrac{(b-a)i}{n})}_{\text{height}}\underbrace{(\tfrac{b-a}{n})}_{\text{width}}\right)$$

Evaluate the limit by recognizing the sum as a Riemann sum for a function defined on [0, 1]

1.) $lim_{n\to\infty}\Sigma_{i=1}^{n} sin(\frac{i}{n})\frac{1}{n}$ where $f(x) = \sin x$

$$= \int_0^1 \sin x \, dx$$
$$= -\cos x \Big|_0^1$$
$$-\cos(1) - (-\cos(0))$$
$$= 1 - \cos(1)$$



2.) $lim_{n\to\infty}\Sigma_{i=1}^{n} \frac{i^5}{n^6}$ where $g(x) = x^5$

$$= \lim_{n\to\infty} \sum_{i=1}^{n}\left(\frac{i^5}{n^5}\right)\left(\frac{1}{n}\right) = \int_0^1 x^5 dx = \frac{x^6}{6}\Big|_0^1 = \frac{1}{6}$$

The Fundamental Theorem of Calculus: Suppose f continuous on $[a, b]$.

$$f \xrightarrow[\text{anti-derivate}]{\text{an}} \int_a^x f(t)dt \xrightarrow{\text{derivative}} f$$

1.) If $G(x) = \int_a^x f(t)dt$, then $G'(x) = f(x)$.
G is an anti-derivative of f

2.) $\int_a^b f(t)dt = F(b) - F(a)$ where F is any antiderivative of f, that is $F' = f$. useful

The Fundamental Theorem of Calculus: Suppose f continuous on $[a, b]$.

take derivative

1.) If $\frac{d}{dx}[\underbrace{\int_a^x f(t)dt}_{\text{take an anti-derivative}}] = f(x)$.

2.) $\int_a^b F'(t)dt = F(b) - F(a)$. $\Leftarrow$ useful

Examples:

1.) If $G_1(x) = \int_0^x t^2 dt$, then $G_1'(x) =$ x^2.

2.) If $G_2(x) = \int_5^x t^2 dt$, then $G_2'(x) =$ x^2.

3.) If $G_3(x) = \int_{-2}^x sin(t^2)dt$, then $G_3'(x) =$ $\sin(x^2)$.

4.) If $G_4(x) = \int_4^x tan(\frac{t^3}{t+1})dt$, then $G_4'(x) =$ $\tan\left(\frac{x^3}{x+1}\right)$.

5.) If $G_5(x) = \int_1^x \sqrt{3t-5}dt$, then $G_5'(x) =$ $\sqrt{3x-5}$.

$$\boxed{G_1(x) = \int_0^x t^2 dt} = \frac{t^3}{3}\Big|_0^x = \boxed{\frac{x^3}{3}} - 0$$

$$\Rightarrow G_1'(x) = x^2$$

$$G_2(x) = \int_5^x t^2 dt = \frac{t^3}{3}\Big|_5^x = \boxed{\frac{x^3}{3} - \frac{5^3}{3} = G_2(x)}$$

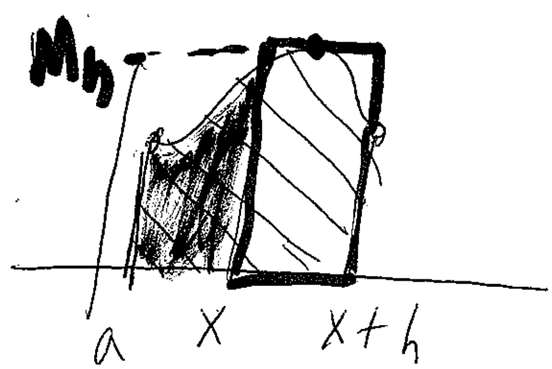
$$\Rightarrow G_2'(x) = x^2$$

The Fundamental Theorem of Calculus: Suppose f continuous on $[a, b]$.

1.) If $G(x) = \int_a^x f(t)dt$, then $G'(x) = f(x)$.

Proof

$$G'(x) = lim_{h\to 0}\frac{G(x+h)-G(x)}{h},$$



$$= lim_{h\to 0}\frac{\int_a^{x+h} f(t)dt - \int_a^x f(t)dt}{h}$$

$$= lim_{h\to 0}\frac{\int_x^{x+h} f(t)dt}{h}$$

$$\leq lim_{h\to 0}\frac{\int_x^{x+h} M_h dt}{h}$$

where $M_h = max\{f(t) \mid x \leq t \leq t+h\}$

(Note M_h exists by extreme value thm)

$$\leq lim_{h\to 0}\frac{(M_h)(\cancel{h})}{\cancel{h}}$$

$$\leq lim_{h\to 0} M_h = f(x)$$

Similarly $G'(x) \geq f(x)$

(using $m_h = min\{f(t) \mid x \leq t \leq t+h\}$

The Fundamental Theorem of Calculus: Suppose f continuous on $[a, b]$.

2.) $\int_a^b f(t)dt = F(b) - F(a)$ where F is any antiderivative of f, that is $F' = f$.

Proof

Let $G(x) = \int_a^x f(t)dt$. Then $G'(x) = f(x)$ (ie, G is an antiderivative of f).

Let F be any antiderivative of f.

Then $F(x) = G(x) + C = \int_a^x f(t)dt + C$ for some constant C.

Thus $F(b) - F(a) = G(b) + C - [G(a) + C]$

$= G(b) - G(a) = \int_a^b f(t)dt - \int_a^a f(t)dt = \int_a^b f(t)dt.$

Find the average of 3, 2, 5, 6:

$$\frac{3 + 2 + 5 + 6}{4} = \frac{16}{4} = 4$$

The average value of n values, $f(t_1), ..., f(t_n)$ is

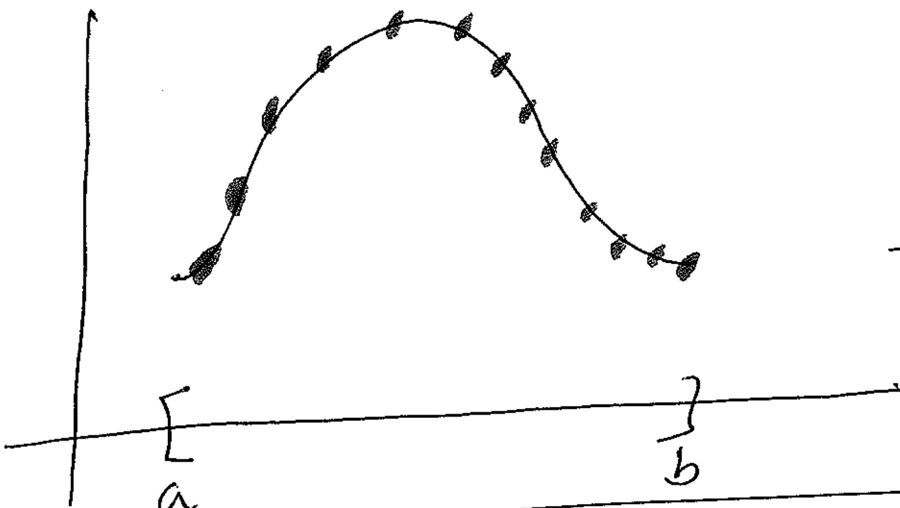
missing

$$\frac{f(t_1) + f(t_2) + ... + f(t_n)}{n} = \frac{\Sigma_{i=1}^n f(t_i)}{n}$$

compare to $\int_a^b f(t)dt = \lim_{n \to \infty} \sum_{i=1}^n f(t_i) \Delta t$

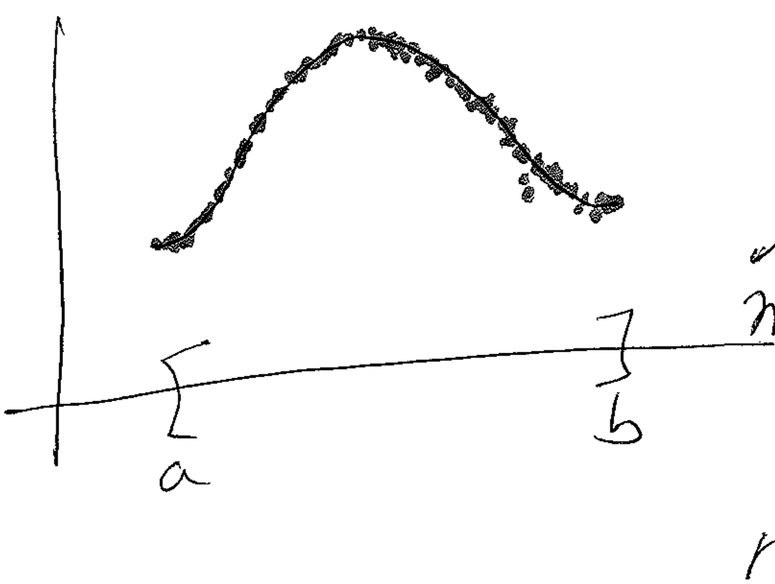
The average of

$f(t_1), \ldots\ f(t_n)$



$$\frac{\sum_{i=1}^{n} f(t_i)}{n}$$

Find the average of f over $[a,b]$

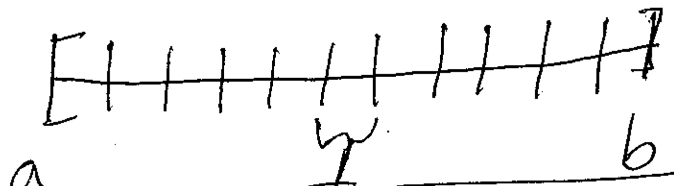


$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} f(t_i)}{n}$$

related to $\int_a^b f(t)\,dt$

BUT width is missing

$$\int_a^b f(t)\,dt = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_i)\Delta t$$



$$\boxed{\Delta t = \frac{b-a}{n}}$$

Average of f over $[a,b]$

$$= \lim_{n\to\infty} \frac{\left(\sum_{i=1}^{n} f(t_i)\right)\Delta t}{n \cdot \Delta t}$$

$$= \lim_{n\to\infty} \left(\frac{1}{(n)(\Delta t)}\right) \sum_{i=1}^{n} f(t_i)\Delta t$$

$$= \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}(b-a)} \sum_{i=1}^{n} f(t_i)\Delta t$$

$$= \frac{1}{b-a}\left[\lim_{n\to\infty} \sum_{i=1}^{n} f(t_i)\Delta t\right] = \frac{1}{b-a}\int_a^b f(t)\,dt$$

The average of f over interval $[a,b]$

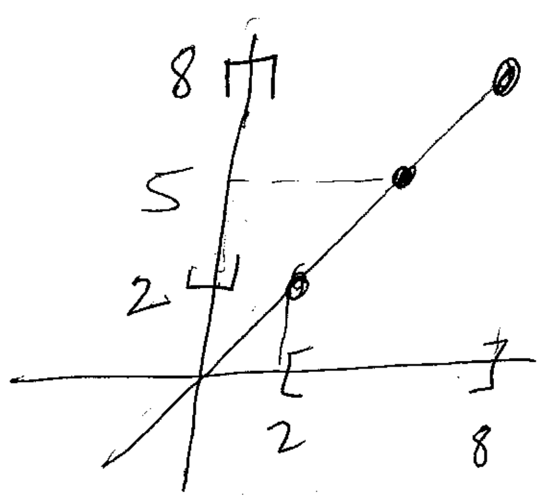
is $\boxed{\frac{1}{b-a}\int_a^b f(t)\,dt}$

Find average of $g(x) = x$ over $[2, 8]$

$$\frac{1}{8-2}\int_2^8 x\,dx$$

$$= \frac{1}{6}\,\frac{x^2}{2}\Big|_2^8 = \frac{1}{6}\left[\frac{64}{2} - \frac{4}{2}\right]$$

$$= \frac{1}{6}[30] = 5$$

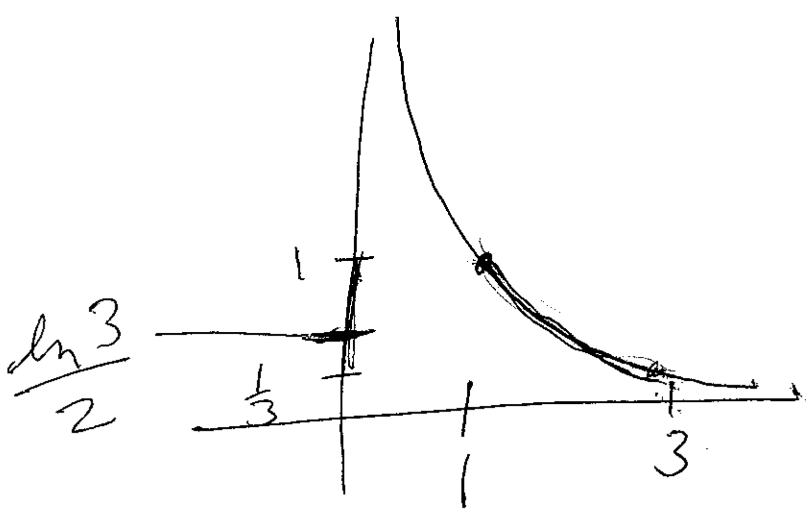


Find average of $h(x) = \frac{1}{x}$ over $[1, 3]$

$$\frac{1}{3-1}\int_1^3 \frac{1}{x}\,dx$$

$$= \frac{1}{2} \ln|x| \Big|_1^3$$

$$= \frac{1}{2}\left[\ln 3 - \ln 1\right] = \frac{\ln 3}{2}$$



Find average of $y = \sin x$ over $[0, \pi]$

$$\frac{1}{\pi}\int_0^{\pi} \sin x \, dx$$

$$= \frac{1}{\pi}\left(-\cos x\right)\Big|_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\cos\pi - (-\cos(0))\right]$$

$$= \frac{1}{\pi}\left[1 + 1\right] = \frac{2}{\pi}$$