Section 5.2

Find the area under the curve $f(x) = -\frac{1}{2}x + 4$, above the $x$–axis and between $x = 2$ and $x = 6$.

Method 1: In this case our function is very simple, so we can determine the area without calculus:

$$\frac{1}{2} BH - \frac{1}{2} bh = \frac{1}{2} (8-2)(3) - \frac{1}{2} (2)(1)$$

$$= \frac{1}{2} 6 \cdot 3 - 1 \cdot 1 = 9 - 1 = 8$$
Method 2: Estimate using rectangles.

Inscribed rectangles with $\Delta x = 1$:

$$6 - 2 = 4$$

Length $= 6 - 2$

$$\frac{4}{4} \Rightarrow \text{# of rectangle}$$

Width $= \frac{4}{4} = 1$

$$\sum_{i=3}^{6} f(x_i) \Delta x = \sum_{i=1}^{\frac{4}{4}} f(i)(1) = \sum_{i=1}^{\frac{4}{4}} f(i + 2)(1)$$

width $= 1$

$$= \left[ \frac{-1}{2} (3) + 4 \right] (1) + \left[ \frac{-1}{2} (4) + 4 \right] (1) + \left[ \frac{-1}{2} (5) + 4 \right] (1) + \left[ \frac{-1}{2} (6) + 4 \right] (1)$$

$$= \frac{5}{2} (1) + 2(1) + \frac{3}{2} (1) + 1(1) = 7 < 8$$

since inscribed
Inscribed rectangles with $\Delta x = \frac{6-2}{n} = \frac{4}{n}$.

If have $n$ rectangles

$\frac{6-2}{n} = \frac{4}{n}$

width $= \frac{4}{n}$.

Sum of the area of rectangles

$= \text{sum of heights} \times \text{widths} = \sum f(x_i) (\frac{4}{n})$

Total area $= \sum f(x_i) \cdot \Delta x$

estimate
Circumscribed rectangles with $\Delta x = 1$:

\[
\sum_{i=1}^{4} f(x_i)(1) = \sum_{i=1}^{4} f(i+1)(1)
\]

\[
f(2)(1) + f(3)(1) + f(4)(1) + f(5)(1) =
\]

\[
= [-\frac{1}{2}(2) + 4](1) + [-\frac{1}{2}(3) + 4](1)
+ [-\frac{1}{2}(4) + 4](1) + [-\frac{1}{2}(5) + 4](1)
\]

\[
= 3 + \frac{5}{2}(1) + 2(1) + \frac{3}{2}(1) = 9 > 8
\]
Estimate the distance traveled between \( t = 2 \) and \( t = 6 \) if the velocity is given by the function \( f(t) = -\frac{1}{2}t + 4 = v(t) \).

Estimate using inscribed rectangles with \( \Delta t = 1 \):

Under estimate of distance traveled = sum of areas of rectangles

\[
f(3)(1) + f(4)(1) + f(5)(1) + f(6)(1) = \\
= \left[ -\frac{1}{2}(3) + 4 \right](1) + \left[ -\frac{1}{2}(4) + 4 \right](1) \\
\text{ } + \left[ -\frac{1}{2}(5) + 4 \right](1) + \left[ -\frac{1}{2}(6) + 4 \right](1) \\
= \frac{5}{2}(1) + 2(1) + \frac{3}{2}(1) + 1(1) = 7 \text{ miles}
\]
\[ \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x = \int_{a}^{b} f(x) \, dx \]

$\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(a + i \Delta x) \Delta x$

If $f$ is continuous, can use inscribed rectangles, circumscribed rectangles, all left-hand endpoints, all right-hand endpoints, or all midpoints, etc.
can use midpts
instead of error
inscribed or circumscribed rectangles

can use all right-hand end pts
Find the distance traveled between \( t = 2 \) and \( t = 6 \) if the velocity is given by the function \( f(t) = -\frac{1}{2} t + 4 \).

= \text{arc } a

Method 1: In this case our function is very simple, so we can determine the area without calculus:

\[
\frac{1}{2} \cdot B \cdot H - \frac{1}{2} \cdot b \cdot h \\
\frac{1}{2} (8-2) (3) - \frac{1}{2} (2) (1) = 9 - 1 = 8
\]

Method 2: Use calculus by estimating with rectangles and taking limit.

\[
\text{Area} = \lim_{n \to \infty} \sum_{i=1}^{n} f\left(a + \frac{(b-a)i}{n}\right) \left(\frac{b-a}{n}\right)
\]
\[
= \lim_{n \to \infty} \sum_{i=1}^{n} f\left(2 + \frac{4i}{n}\right) \left(\frac{4}{n}\right)
\]
\[
= \lim_{n \to \infty} \sum_{i=1}^{n} \left[-\frac{1}{2} \left(2 + \frac{4i}{n}\right) + 4\right] \left(\frac{4}{n}\right) = 8
\]

Method 3 (section 5.3): Use calculus by integrating.

\[
\int_{2}^{6} \left(-\frac{1}{2} t + 4\right) dt = \left(-\frac{1}{4} t^2 + 4t\right)|_{2}^{6}
\]
\[
= \left(-\frac{1}{4} (6)^2 + 4(6)\right) - \left(-\frac{1}{4} (2)^2 + 4(2)\right)
\]
\[
= -9 + 24 - (-1 + 8) = 15 - 7 = 8 \checkmark
\]
Example:

\[ \int_2^6 \left( -\frac{1}{2}t + 4 \right) dt = \lim_{n \to \infty} \sum_{i=1}^{n} f(t_i) \Delta t \]

\[ \Delta t = \frac{6-2}{n} = \frac{4}{n} \] (using \( n \) equal subintervals)

\[ t_i = 2 + i \Delta t = 2 + \frac{4i}{n} \] (using right-hand endpoints)

\[ \int_2^6 \left( -\frac{1}{2}t + 4 \right) dt = \lim_{n \to \infty} \sum_{i=1}^{n} f \left( 2 + \frac{4i}{n} \right) \left( \frac{4}{n} \right) \]

\[ = \lim_{n \to \infty} \sum_{i=1}^{n} \left[ -\frac{1}{2} \left( 2 + \frac{4i}{n} \right) + 4 \right] \left( \frac{4}{n} \right) \]

\[ = \lim_{n \to \infty} \sum_{i=1}^{n} \left[ -1 - \frac{2i}{n} + 4 \right] \left( \frac{4}{n} \right) \]

\[ = \lim_{n \to \infty} \sum_{i=1}^{n} \left[ 3 - \frac{2i}{n} \right] \left( \frac{4}{n} \right) \]

\[ = \lim_{n \to \infty} \sum_{i=1}^{n} \left[ \frac{12}{n} - \frac{8i}{n^2} \right] \]

\[ = \lim_{n \to \infty} \left( \sum_{i=1}^{n} \frac{12}{n} - \sum_{i=1}^{n} \frac{8i}{n^2} \right) \]

\[ = \lim_{n \to \infty} \left( 12 - \frac{8}{n^2} \sum_{i=1}^{n} i \right) \]

\[ = \lim_{n \to \infty} \left( 12 - \frac{8}{n^2} \frac{n(n+1)}{2} \right) \]

\[ = \lim_{n \to \infty} \left( 12 - \frac{4n^2+4n}{n^2} \right) \]

\[ = \lim_{n \to \infty} \left( 12 - 4 - \frac{4}{n} \right) = 8 \]
\[ \sum_{i=1}^{100} i = \frac{10100}{2} \]

\[ = 1 + 2 + 3 + \ldots + 98 + 99 + 100 \]
\[ + 100 + 99 + 98 + \ldots + 3 + 2 + 1 \]
\[ = \frac{101 + 101 + 101 + \ldots + 101 + 101 + 101}{101} \]
\[ = (100)(101) = 10100 \]

\[ \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \]