Let \( N(d_1, ..., d_n) \) be the number of labeled trees with \( n \) vertices \( \{v_1, ..., v_n\} \) such that \( \text{deg}(v_i) = d_i + 1 \).

Let \( C(n-2, d_1, ..., d_n) = \frac{(n-2)!}{d_1!d_2!...d_n!} \).

section 3.5 34e.) Claim: \( N(d_1, ..., d_n) = \begin{cases} C(n-2; d_1, ..., d_n) & \text{if } \Sigma d_i = n-2 \\ 0 & \text{otherwise} \end{cases} \)

Claim \( N(d_1, ..., d_n) = 0 \) if \( \Sigma d_i \neq n-2 \)

Proof: \( \Sigma d_i = \Sigma_{i=1}^{n}(\text{deg}(v_i) - 1) = [\Sigma_{i=1}^{n}(\text{deg}(v_i))] - n = [2(n-1)] - n = 2n - 2 - n = n - 2 \)

Thus \( \Sigma d_i = n - 2 \). Hence \( N(d_1, ..., d_n) = 0 \) if \( \Sigma d_i \neq n-2 \).

Claim \( N(d_1, ..., d_n) = C(n-2; d_1, ..., d_n) \) if \( \Sigma d_i = n-2 \) \hspace{1cm} (*)

Proof by induction on \( k \) = number of vertices.

By part a, the equality holds for \( n = 2 \).

Induction hypothesis: Suppose (*) is true when \( k = n - 1 \).

By part b, \( d_j = 0 \) for some \( j \). WLOG assume \( j = n \). Thus by part c,

\[ N(d_1, ..., d_n) = N(d_1, ..., d_{n-1}, 0) = N(d_1-1, d_2, ..., d_{n-1}) + N(d_1, d_2-1, d_3, ..., d_{n-1}) + ... + N(d_1, ..., d_{n-2}, d_{n-1} - 1). \]

By part d,

\[ C(n-2; d_1, ..., d_n) = C(n-2; d_1, ..., d_{n-1}, 0) = C(n-3; d_1-1, d_2, ..., d_{n-1}) + C(n-3; d_1, d_2-1, d_3, ..., d_{n-1}) + ... + C(n-3; d_1, ..., d_{n-2}, d_{n-1} - 1). \]

By the induction hypothesis, \( N(d_1, ..., d_n) = N(d_1, ..., d_{n-1}, 0) = N(d_1-1, d_2, ..., d_{n-1}) + \)
\( N(d_1, d_2-1, d_3, ..., d_{n-1}) + ... + N(d_1, ..., d_{n-2}, d_{n-1} - 1) = C(n-3; d_1-1, d_2, ..., d_{n-1}) + \)
\( C(n-3; d_1, d_2-1, d_3, ..., d_{n-1}) + ... + C(n-3; d_1, ..., d_{n-2}, d_{n-1} - 1) = C(n-2; d_1, ..., d_{n-1}, 0) = C(n-2; d_1, ..., d_n) \).

34a). Suppose \( n = 2 \). A tree with 2 vertices has 1 edge. Thus \( \text{deg}(v_i) = 1 \) for \( i = 1, 2 \). Thus \( d_i = 0 \) for \( i = 1, 2 \). There is exactly one labeled tree with 2 vertices, \( T = \{\{v_1, v_2\}\} \).
Thus \( N(0,0) = 1 \). \( C(0;0,0) = \frac{0!}{0!0!} = 1 \). Thus (*) holds for \( n = 2 \).

34b.) Claim \( d_i = 0 \) for some \( i \).

Suppose \( d_i > 0 \) for all \( i \). Then \( \deg(v_i) = d_i + 1 > 1 \) for all \( i \). That is, \( \deg(v_i) \geq 2 \) for all \( i \).

The number of edges in a graph \( = \frac{1}{2} \sum \deg(v_i) \).

The number of edges in a tree with \( n \) vertices is \( n - 1 \).

Thus \( n - 1 = \frac{1}{2} \sum_{i=1}^{n} \deg(v_i) \geq \frac{1}{2} \sum_{i=1}^{n} 2 = \frac{1}{2} (2n) = n \), a contradiction. Thus \( d_i = 0 \) for some \( i \).

34c). Let \( A = \) set of all labeled trees with \( n \) vertices \( \{v_1, ..., v_n\} \) such that \( \deg(v_i) = d_i + 1 \).

Then \( |A| = N(d_1, ..., d_n) \).

For \( j = 1, ..., n-1 \), let \( B_j = \) set of all labeled trees with \( n-1 \) vertices \( \{v_1, ..., v_{n-1}\} \) such that \( \deg(v_i) = d_i + 1, i \neq j \) and \( \deg(v_j) = (d_j - 1) + 1 \).

Then \( |B_j| = N(d_1, ..., d_{j-1}, d_j - 1, d_{j+1}, ..., d_{n-1}) \) for \( j = 1, ..., n - 1 \).

Note if \( T_j \in B_j \), then \( \deg(v_j) = d_j \).

Suppose \( k \neq j \). If \( T_k \in B_k \), then \( \deg(v_j) = d_j + 1 \). Thus \( T_j \) is not isomorphic to \( T_j \).

Thus \( B_j \cap B_k = \emptyset \) for \( k \neq j \).

Claim: There exists a bijection \( f : A \to \bigcup_{i=1}^{n-1} B_i \).

Note if the claim is true, then \( |A| = |\bigcup_{i=1}^{n-1} B_i| = \sum_{i=1}^{n-1} |B_i| \), since the \( B_i \) are pairwise disjoint.

Define \( g : \bigcup_{i=1}^{n-1} B_i \to A \). Let \( T = (V, E) \in B_j \). Let \( g(T) = (V \cup \{v_n\}, E \cup \{v_j, v_n\}) \).

Note \( g(T) \) has \( n \) vertices and \( \deg(v_i) = d_i + 1 \) for \( i = 1, ..., n \). Thus \( g : \bigcup_{i=1}^{n-1} B_i \to A \) is well-defined.

Claim \( g^{-1} \) exists.
WLOG assume $d_n = 0$ (relabel the vertices if needed). $d_n = 0$ implies $\deg(v_n) = 1$. Suppose the vertex adjacent to $v_n$ is labeled $v_j$. Let $T'(V', E') \in \mathcal{A}$. Define $f : A \rightarrow \bigcup_{i=1}^{n-1} B_i$ by $f(T') = (V' - \{v_n\}, E' - \{\{v_j, v_n\}\})$. Note that $f(T')$ has $n - 1$ vertices, $\{v_1, \ldots, v_{n-1}\}$ and $\deg(v_j) = d_j$, $\deg(v_i) = d_i + 1$ for $i \neq j$. Thus $f(T')$ is in $B_j$, and hence $f$ is well-defined.

$$f(g((V, E)))) = f((V \cup \{v_n\}, E \cup \{\{v_j, v_n\}\})) = (V, E).$$

$$g(f((V, E))) = g((V - \{v_n\}, E - \{\{v_j, v_n\}\})) = (V, E).$$

Thus $g$ is invertible. Thus $g$ is a bijection. Thus $|A| = |\bigcup_{i=1}^{n-1} B_i| = \sum_{i=1}^{n-1} |B_i|$. Alternate proof that $g$ is a bijection:

Claim: $g$ is onto. Let $T' = (V', E') \in \mathcal{A}$. Let $T = (V' - \{v_n\}, E' - \{\{v_j, v_n\}\})$. Then $g(T) = g((V' - \{v_n\}, E' \cup \{\{v_j, v_n\}\})) = (V', E') = T'$.

Claim $g$ is 1-1:

Suppose $g(T) = g(S)$. Claim $T$ and $S$ are isomorphic ...

34d). Note that by the right-hand side of the equation, we are given that

$$\sum_{i=1}^{n-1} d_i = [\sum_{i=1}^{n} d_i] - d_n = n - 2 - 0 = n - 2$$

$$\sum_{i=1}^{n-1} C(n - 3; d_1, \ldots, d_{i-1}, d_i - 1, d_{i+1}, \ldots, d_{n-1}) = \sum_{i=1}^{n-1} \frac{(n-3)!}{d_1! \cdots d_{i-1}! (d_i-1)! (d_{i+1}! \cdots d_{n-1}!)}$$

$$= \sum_{i=1}^{n-1} \frac{(n-3)!}{d_1! \cdots d_{i-1}! (d_i-1)! d_{i+1}! \cdots d_{n-1}!} \sum_{i=1}^{n-1} d_i$$

$$= \frac{(n-3)!}{d_1! \cdots d_{n-1}!} (n - 2)$$

$$= \frac{(n-2)!}{d_1! \cdots d_{n-1}!} = \frac{(n-2)!}{d_1! \cdots d_{n-1}! d_n} = C(n - 2, d_1, \ldots, d_n)$$