1.) Answer will be provided during Tuesday's review session.

2.) Find the number of necklaces you can create containing 4 beads if you have exactly 20 beads and each bead is unique?

$$\frac{(20)(19)(18)(17)}{(4)(2)}$$

Note (20)(19)(18)(17) is the number of 4-permutations of 20 distinct objects. We divide by 4 since we can rotate the necklace by 0, 90, 180, or 270 degrees. We divide by 2 since we can turn over the necklace.

3.) Find the number of necklaces you can create containing 4 beads if you have exactly 1 red bead, 1 blue bead, and two identical green beads.

Place red bead first. Then place blue bead. Note there are only two distinct choices for the blue bead since we can turn the necklace over. Place green beads in the remaining spots.

Thus there are 2 such necklaces.

Notice that this reasoning worked because we had very few beads and we used ALL the beads. When things get more complicated, use Thm 14.2.3. However, when possible, simplify first as in the next problem.

4.) Find the number of necklaces you can create containing 21 beads if the necklace must contain 1 red bead, any number of yellow beads or blue beads.

We could make the following definitions:

Let $X = \{1, 2, ..., 21\}$.

Let $\mathcal{C} = \{f : X \to \{$ red, yellow, blue $\mid \exists$ a unique i such that $f(i) =$ red $\}$

Let $G = \{id, \rho, \rho^2, ..., \rho^{21}, \tau_1, ..., \tau_{21}\}$ where ρ^i is a rotation and τ_i is a reflection.

However, we can instead simplify the problem first.

Since we must use exactly 1 red bead, place the red bead first. Now we do not need to mod out by rotations. Since we don't want to change the location of the red bead, we only need to mod out by a single reflection. Also, we only need to count the number of ways to place 20 beads, some of which are blue and some of which are yellow. Thus:

Let $X = \{1, 2, ..., 20\}$.

Let $\mathcal{C} = \{f : X \to \{$ yellow, blue $\}$

Let $G = \{id, \tau_1\}$ where ρ^i is a rotation and τ_i is the reflection which does not change the position of the red bead.

$N(G, \mathcal{C}) = \frac{1}{|G|}\Sigma_{f\in G}|\mathcal{C}(f)|$ where $\mathcal{C}(f) = \{\mathbf{c} \in \mathcal{C} \mid f * \mathbf{c} = \mathbf{c}\}$.

$|\mathcal{C}(id)| = 2^{20}$ since the identity function fixes all colorings.

$|\mathcal{C}(\tau_1)| = 2^{10}$ by the following: Color the first 10 beads starting from the first bead to the right of the red bead. There are 2^{10} ways to color these 10 beads blue and yellow. Since τ_1 preserves the bead coloring, the colors of the remaining 10 beads are determined by first 10 beads.

Thus $N(G, \mathcal{C}) = \frac{1}{2}(2^{20} + 2^{10}) = 2^{19} + 2^9$

5.) Suppose a code is created using the numbers $\{0,1,2\}$. Suppose the code must contain an even number of 0's. How many different codes of length n can be created?

Method 1: exponential generating function (**we didn't cover this shorter method** – it is from section 7.3. It is very similar to section 7.2, but works for permutations instead of combinations):

$g(x) = (\frac{x^0}{0!} + \frac{x^2}{2!} + \frac{x^4}{4!} + ...)(\frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + ...)(\frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + ...) = (\frac{e^x+e^{-x}}{2})(e^x)(e^x) = \frac{e^{3x}+e^x}{2} = \frac{1}{2}[\Sigma_{n=0}^{\infty} 3^n \frac{x^n}{n!} + \Sigma_{n=0}^{\infty} \frac{x^n}{n!}] = \Sigma_{n=0}^{\infty}(\frac{3^n+1}{2})\frac{x^n}{n!}$

Thus the number of different codes of length n is $\frac{3^n+1}{2}$

Method 2: recurrence relation.

Let h_n = the number of different codes of length n.

Suppose the code is $x_1x_2...x_n$.

Case 1: $x_n = 1, 2$

In this case $x_1x_2...x_{n-1}$ contains an even number of 0's and is thus a code of length $n-1$. By definition h_{n-1} = the number of codes of length $n-1$.

Thus the number of codes of the form $x_1x_2...x_n$ where $x_n = 1, 2$ is $2h_{n-1}$ since there are h_{n-1} choices for $x_1x_2...x_{n-1}$ and 2 choices for x_n.

Case 2: $x_n = 0$.

Then $x_1x_2...x_{n-1}$ is a sequence of length $n-1$ which contains an odd number of 0's.

The number of sequences of length $n-1$ containing 0, 1, 2 with no restrictions is 3^{n-1}

The number of sequences of length $n-1$ containing 0, 1, 2 which contains an even number of 0's is h_{n-1}

Thus the number of sequences of length $n-1$ containing 0, 1, 2 which contains an odd number of 0's is $3^{n-1} - h_{n-1}$

From cases 1 and 2 we obtain:

h_n = the number of codes of length $n = 2h_{n-1} + 3^{n-1} - h_{n-1} = h_{n-1} + 3^{n-1}$

We need initial conditions. We can use either $h_0 = 1$ since the empty code is the only code of length 0 or if you prefer, you can use $h_1 = 2$ (since valid codes of length one are 1 and 2)

To solve recurrence relation: $h_n = h_{n-1} + 3^{n-1}$, $h_0 = 1$.

1.) Solve homogeneous recurrence relation: $h_n = h_{n-1}$.

Guess $h_n = q^n$. Then $h_n = h_{n-1}$ implies $q^n = q^{n-1}$. Hence $q = 1$.

Thus the general solution to the homogeneous recurrence relation is $c(q^n) = c(1^n) = c$.

Note the general solution for a homogeneous recurrence relation is always a linear combination of specific solutions. If the recurrence relation is first order (ie only involves h_n and h_{n-1}), then the general solution for a LINEAR HOMOGENEOUS recurrence relation will be of the form cq^n

for some q (note q can be any complex number). If the recurrence relation is second order (ie only involves h_n, h_{n-1}, and h_{n-2}), then the general solution for a LINEAR HOMOGENEOUS recurrence relation will be of the form $c_1q_1^n + c_2q_2^n$ for some q_i.

2.) Guess a solution to the non-homogeneous recurrence relation $h_n = 3^{n-1} + h_{n-1}$. Note we only need one solution for the non-homogeneous recurrence relation.

Try a multiple of 3^n. Suppose $h_n = a(3^n)$.

Then $h_n = h_{n-1} + 3^{n-1}$ implies $a(3^n) = 3^{n-1} + a(3^{n-1})$

Thus $3a(3^{n-1}) = (1 + a)(3^{n-1})$

Hence $3a = 1 + a$

$2a = 1$. Hence $a = \frac{1}{2}$. Thus a solution to the non-homogeneous recurrence relation is $h_n = \frac{1}{2}(3^n)$

HENCE, the general solution to the non-homogeneous recurrence relation is $h_n = c + \frac{1}{2}(3^n)$

3.) Use initial conditions to find c:

$h_0 = 1$ implies $1 = c + \frac{1}{2}(3^0)$. Hence $c = \frac{1}{2}$.

Thus the number of different codes of length n is $h_n = \frac{1}{2} + \frac{1}{2}(3^n) = \frac{3^n+1}{2}$

6a.) Suppose someone claims that they used their computer for a total of 241 hours over a 10 day period. Do you believe this person?

Answer: No, there are only 24 hours in a day and thus only 240 hours in 10 days.

6b.) Suppose this person lives on a different planet. Suppose the time it takes for this planet to rotate 360 degrees around its axis corresponds to an integral number of hours as measured in earth time. Suppose also that each day the person uses their computer for an integral number of hours for a total of 241 hours over a 10 day period. What is the minimum possible time it takes for this planet to rotate 360 degrees around its axis.

Answer: 25 hours. Note that if the wording of the problem didn't confuse you, the answer 25 may have seemed obvious. The strange wording is so that we can use the pigeonhole principle which involves non-negative integers. We need to partition the 241 hours into 10 days. The days will correspond to our boxes, the hours will be our objects. We can use the formula in pigeonhole principle-strong form: 10(r-1) + 1 = 241, to determine $r = 25$. Thus there is at least one box containing 25 hours. Thus a day on this planet must be at least 25 hours long. This was a complicated way to determine the answer.

The point of this problem is that you can use common sense. Sometimes the math language can make things more complicated (e.g., "integral number of hours" was needed to use PHP), but think about what makes sense. Then try to explain it. Often math language is helpful, but don't let it get in the way. But don't ignore math notation either. Often math notation tells us how to do the problem. For example in problem 7, PHP is very helpful. Many people would find #7 to be quite hard without having the framework of the PHP to work with.

The next problem uses PHP-weak form: If you have $n + 1$ objects and n boxes, one box has at least two objects. This may be the simplest theorem that we have stated. But note stating it, allows us to prove a variety of very interesting theorems. Having the formal statement involving boxes and objects makes us think that we need to look at determining what the boxes and what the objects could be.

7.) Suppose S is a set of 6 integers. Show that there exists $x, y \in S$ such that $x \neq y$, but $x - y$ is a multiple of 5.

Proof: We will use the pigeonhole principle. Let $S = \{x_1, x_2, x_3, x_4, x_5, x_6\}$. Let r_i be the remainder when x_i is divided by 5 for $i = 1, ..., 6$. Then $r_i \in \{0, 1, 2, 3, 4\}$. Since there are 6 r_i's and 5 possible values for r_i, by the pigeonhole principle there exists $j \neq k$ such that $r_j = r_k$. Thus $x_j = 5n + r_j$ and $x_k = 5m + r_k = 5m + r_j$ for some integers n, m. Thus $x_j - x_k = 5n + r_j - (5m + r_j) = 5(n - m)$.

2nd proof similar to the above, but with explanation: It often helps to be specific. For example instead of just knowing S is a set of 6 integers, I let $S = \{x_1, x_2, x_3, x_4, x_5, x_6\}$. Thus I could more easily work with these 6 elements. Since this is a pigeonhole problem, I looked for a way to define boxes. Since we needed to find x, y such that $x - y$ is a multiple of 5, we will try boxes based upon remainder of $x_i \in S$ when divided by 5. Ie, box 0 is the box containing elements of S which have remainder 0 when divided by 5 (i.e., are multiples of 5), box 1 is the box containing elements of S which have remainder 1 when divided by 5, ... , box 4 is the box containing elements of S which have remainder 4 when divided by 5. Thus there are 5 boxes. Since S has 6 elements, we know that one of the boxes contains at least two elements of S. Suppose x_j and x_k are two elements of S which belong to the same box. Hence they have the same remainder when divided by 5. We can again be specific by using the definition of remainder. Note it often helps to get more specific by using a definition. Thus $x_j = 5n + r_j$ and $x_k = 5m + r_j$ for some integers n, m. Thus $x_j - x_k = 5n + r_j - (5m + r_j) = 5(n - m)$.

Note how to define the boxes is often not obvious, so you may have to try several possibilities. Try many things and don't give up. What doesn't work can give you ideas as to what might work (and can earn you partial credit if you run out of time).

8.) In how many ways can 12 indistinguishable apples and 2 oranges be distributed among three children in such a way that each child gets at least 2 pieces of fruit?

Distribute oranges first:

Suppose first child gets 2 oranges.

Distributing 12 indistinguishable apples among 3 distinguishable children so that each child gets at least two pieces of fruit = number of solutions to $x_1 + x_2 + x_3 = 12$ such that $x_1 \geq 0$, $x_2 \geq 2$, $x_3 \geq 2$ Let $y_1 = x_1 \geq 0$, $y_2 = x_2 - 2 \geq 0$, $y_3 = x_3 - 2 \geq 0$.

Then the number of solutions to $x_1 + x_2 + x_3 = 12$ such that $x_1 \geq 0$, $x_2 \geq 2$, $x_3 \geq 2$ is the same as the number of solutions to $y_1 + y_2 + 2 + y_3 + 2 = 12$ such that $y_1 \geq 0$, $y_2 \geq 0$, $y_3 \geq 0$ which is the same as the number of solutions to $y_1 + y_2 + y_3 = 8$ such that $y_1 \geq 0$, $y_2 \geq 0$, $y_3 \geq 0 = \binom{8+3-1}{8}$

$= \binom{10}{8} = 5(9) = 45$

Since any of the three children could have received the 2 oranges, the number of ways to distribute 2 oranges to one child and 12 indistinguishable apples among 3 distinguishable children so that each child gets at least two pieces of fruit = 3(45)

Suppose exactly one child doesn't get an orange (i.e., two of the children receive exactly one orange):

Distributing 12 indistinguishable apples among 3 distinguishable children so that each child gets at least two pieces of fruit = number of solutions to $x_1+x_2+x_3 = 12$ such that $x_1 \geq 1$, $x_2 \geq 1$, $x_3 \geq 2$ Let $y_1 = x_1 - 1 \geq 0$, $y_2 = x_2 - 1 \geq 0$, $y_3 = x_3 - 2 \geq 0$.

Then the number of solutions to $x_1 + x_2 + x_3 = 12$ such that $x_1 \geq 1$, $x_2 \geq 1$, $x_3 \geq 2$ is the same as the number of solutions to $y_1+1+y_2+1+y_3+2 = 12$ such that $y_1 \geq 0$, $y_2 \geq 0$, $y_3 \geq 0$. which is the same as the number of solutions to $y_1+y_2+y_3 = 8$ such that $y_1 \geq 0$, $y_2 \geq 0$, $y_3 \geq 0$. $= \binom{8+3-1}{8}$ $= \binom{10}{8} = 5(9) = 45$

Since any of the three children could have been the one to not receive an orange, the number of ways to distribute 2 oranges and 12 indistinguishable apples among 3 distinguishable children so that each child gets at least two pieces of fruit and exactly one child doesn't get an orange = 3(45)

Hence the number of ways to distribute 2 oranges and 12 indistinguishable apples among 3 distinguishable children = 3(45) + 3(45) = 270.

9.) Use combinatorial reasoning to prove that $\sum_{k=0}^{n} k(k-1)\binom{n}{k} = n(n-1)2^{n-2}$.

Suppose we wish to form a committee of arbitrary size which includes a chair and a vice-chair. There are $n(n-1)$ ways to choose a chair and a vice-chair from n people. There are 2^{n-2} ways to form the rest of the committee from the remaining $n-2$ people if the committee can have an arbitrary number of people.

Thus the number of different ways to form a committee of arbitrary size which includes a chair and a vice-chair starting with n people is $n(n-1)2^{n-2}$.

Suppose we wish to form a committee of consisting of k people which includes chair and a vice-chair. There are $\binom{n}{k}$ ways to form a committee of k people if we have n people from which to choose. Having formed this committee, there are now $k(k-1)$ ways to choose a chair and a vice-chair from these k people. Thus there are $k(k-1)\binom{n}{k}$ ways to form a committee of consisting of k people which includes a chair and a vice-chair given n people from which to choose.

If the committee can be of arbitrary size then there could be 0 people, 1 person, 2 people, ..., or n people on the committee. Hence the number of different ways to form a committee of arbitrary size which includes a chair and a vice-chair starting with n people is $\Sigma_{k=0}^{n} k(k-1)\binom{n}{k}$.

Thus $\Sigma_{k=0}^{n} k(k-1)\binom{n}{k} = n(n-1)2^{n-2}$.

10.) Suppose Ann, Beth, Carl, Don, and Edna are to be assigned jobs 1, 2, 3, 4, 5. Suppose Ann is qualified for jobs 1, 2, 3; Beth is qualified for jobs 4, 5; Carl is qualified for jobs 1, 4, 5; Don is qualified for jobs 1, 2, 3; and Edna is qualified for jobs 1, 2, 4, 5. How many different job assignments are possible if each person is assigned exactly one job for which they are qualified?

Create a chessboard where columns correspond to Ann, Beth, Carl, Don, and Edna and rows correspond to jobs forbidden to them. Place Ann and Carl in the last two columns so that you can partition the chessboard into two sets A (containing 6 forbidden positions) and B (containing 4 forbidden positions).

Let r_1 = number of ways to place one rook in a forbidden position = number of forbidden positions = 10.

if 1 rook in A: 6

if 1 rook in B: 4

Let r_2 = number of ways to place two rooks in a forbidden positions: 33

if 2 rooks in A: $3 + 2 + 1 + 2 = 7$

if 1 rook in A, 1 in B: $6(4) = 24$

if 2 rooks in B: 2

Let r_3 = number of ways to place three rooks in a forbidden positions: 45

if 3 rooks in A: 1

if 2 rooks in A, 1 in B: $8(4) = 32$

if 1 rook in A, 2 in B: $6(2) = 12$

Let r_4 = number of ways to place four rooks in a forbidden positions: 20

if 3 rooks in A, 1 in B: $1(4) = 4$

if 2 rook in A, 2 in B: $8(2) = 16$

Let r_5 = number of ways to place five rooks in a forbidden positions: 2

if 3 rooks in A, 2 in B: $1(2) = 2$

Hence by thm 6.4.1, the number of different assignments is

$5! - r_1 4! + r_2 3! - r_3 2! + r_4 1! - r_5 0! = 5! - 10(4!) + 33(3!) - 45(2!) + 20 - 2$