Solve $ay'' + by' + cy = g(t)$, $y(0) = y_0$, $y'(0) = y_1$

Step 1: Solve homogeneous eqn $ay''+by'+cy = 0$ (**) $\blacksquare$

Step 1a: Guess solution to (**): Suppose $y = e^{rt}$

$y = e^{rt}$ implies $y' = re^{rt}$ and $y'' = r^2e^{rt}$

$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$ implies $ar^2 + br + c = 0$,

Thus $r = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

Let $r_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $r_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$.

Thus $y = e^{r_1t}$ and $y = e^{r_2t}$ are both solutions to (**)

We will assume $r_1 \neq r_2$ so that we have two different solutions.

Step 1b: Find general soln to homogeneous eqn (**) $\blacksquare$

Note that (**) is a linear equation. Thus since
$y = e^{r_1t}$ and $y = e^{r_2t}$ are both solutions to (**),
$y = c_1e^{r_1t} + c_2e^{r_2t}$ is the general solution to (**).

Step 2: Solve $ay'' + by' + cy = g(t)$ (*)

Step 2a: Guess solution to (*).
Step 2b: Use thm below to form general soln to (*).

Thm: Suppose $c_1\phi_1(t) + c_2\phi_2(t)$ is a general solution to

$$ay'' + by' + cy = 0,$$

If $y(t) = \psi(t)$ is a solution to

$$ay'' + by' + cy = g(t) \ [*],$$

Then $\psi(t)+c_1\phi_1(t)+c_2\phi_2(t)$ is also a solution to [*].

Step 3: If initial value problem:

Once general solution is known, can solve initial value problem (i.e., use initial conditions to find c_1, c_2):

General solution: $y(t) = \psi(t) + c_1\phi_1(t) + c_2\phi_2(t)$
Initial conditions: $y(0) = y_0$, $y'(0) = y_1$

Solve the following system of eqns for c_1 and c_2:

$y_0 = \psi(0) + c_1\phi_1(0) + c_2\phi_2(0)$
$y_1 = \psi'(0) + c_1\phi_1'(0) + c_2\phi_2'(0)$