Thm 2.2.1 Pigeonhole Principle (strong form): Let \( q_1, q_2, \ldots, q_n \) be positive integers. If \( q_1 + q_2 + \ldots + q_n - n + 1 \) objects are put into \( n \) boxes, then for some \( i \) the \( i \)th box contains at least \( q_i \) objects.

Cor: If \( q_i = r \) for all \( i \), then if \( n(r - 1) + 1 \) objects are put into \( n \) boxes, then there exists a box containing at least \( r \) objects.

Cor: If \( \frac{m_1 + \ldots + m_n}{n} > r - 1 \), then there exists an \( i \) such that \( m_i \geq r \).

Appl 9: Show that every sequence \( a_1, a_2, \ldots, a_{n^2+1} \) contains either an increasing or decreasing subsequence of length \( n+1 \).

Example (\( n = 2 \)):

\( a_1 = 8, a_2 = 4, a_3 = 10, a_4 = 6, a_5 = 4 \)

Need \( n + 1 \) objects in our subsequence. Suppose \( r = n + 1 \).

Hence might need \( n(r - 1) + 1 = n(n + 1 - 1) + 1 = n^2 + 1 \) objects in \( n \) boxes in order to obtain at least \( r = n + 1 \) objects in one of the boxes.

Let \( m_k = \) length of largest increasing subsequence beginning with \( a_k \).

\[
\begin{align*}
8 & \quad 8, 10 & \quad m_1 = 2 \\
4 & \quad 4, 10 & \quad 4, 6 & \quad 4, 4 & \quad m_2 = 2 \\
10 & \quad m_3 = 1 & \quad 6 & \quad m_4 = 1 & \quad 4 & \quad m_5 = 1
\end{align*}
\]
Proof: Let $m_k =$ length of largest increasing subsequence beginning with $a_k$, $k = 1, \ldots, n^2 + 1$.

Suppose there exists an $m_k \geq n + 1$. Then there exists an increasing subsequence of length $m_k \geq n + 1$. Hence there exists an increasing subsequence of length $n + 1$.

Suppose $m_k < n + 1$. Then $m_k = 1, 2, \ldots, n$.

Hence there exists an $i$ such that $m_k = i$ for $n + 1$ $a_k$’s.

There exists $a_{k_1}, a_{k_2}, \ldots, a_{k_{n+1}}$ such that

$$m_{k_1} = m_{k_2} = \ldots = m_{k_{n+1}} = i$$

Show $a_{k_1}, a_{k_2}, \ldots, a_{k_{n+1}}$ is a decreasing sequence.

Suppose not. Then there exists a $j$ such that $a_{k_j} > a_{k_{j+1}}$.

$\exists$ an increasing subsequence of length $i$ starting at $a_{k_j}$

There does not exist an increasing subsequence of length $i + 1$ starting at $a_{k_j}$

$\exists$ an increasing subsequence of length $i$ starting at $a_{k_{j+1}}$

There does not exist an increasing subsequence of length $i + 1$ starting at $a_{k_{j+1}}$

Suppose $a_{k_{j+1}}, a_{h_2}, a_{h_3}, \ldots, a_{h_i}$ is an increasing subsequence of length $i$.

Then $a_{k_j}, a_{k_{j+1}}, a_{h_2}, a_{h_3}, \ldots, a_{h_i}$ is an increasing subsequence of length $i + 1$, a contradiction.