Math 150 Exam 2 Form C
October 28, 2015

[3] 1.) In the expansion of $(2x + 5y - z - 1)^{10}$, the coefficient of $x^4y^3z^5$ is 0.

[5] In the expansion of $(2x + 5y - z - 1)^{10}$, the coefficient of x^2yz is $-\frac{10(10!)}{6!}$.

$$\frac{10!}{2!1!1!6!}(2x)^2(5y)(-z)(-1)^6 = -\frac{10(10!)}{6!}x^2yz$$

[13] 2.) Let $S = \{x_6, x_5, x_4, x_3, x_2, x_1, x_0\}$.

What subset of S corresponds to 1101101? $\{x_6, x_5, x_3, x_2, x_0\}$

$1101101 = 2^6 + 2^5 + 2^3 + 2^2 + 2^0$

What subset comes before the subset $\{x_4\}$? $\{x_3, x_2, x_1, x_0\}$

$\{x_4\}$ corresponds to 2^4 and $2^4 = 10000$.

$$\begin{array}{r} 10000 \\ -\ 1 \\ \hline 01111 \end{array}$$

What subset comes after the subset $\{x_4\}$? $\{x_4, x_0\}$

$$\begin{array}{r} 10000 \\ +\ 1 \\ \hline 10001 \end{array}$$

[10] 3.) How many permutations of $\{1, 2, 3, 4, 5, 6, 7\}$

A permutation corresponds to inversion sequence $a_1, ..., a_7$ where a_i are non-negative integers and $a_1 \leq 6$, $a_2 \leq 5$, $a_3 \leq 4$, $a_4 \leq 3$, $a_5 \leq 2$, $a_6 \leq 1$, $a_7 = 0$.

$6 + 5 + 4 + 3 + 2 + 1 + 0 = 21$

[4] a.) have exactly 20 inversions? 6

The only inversion sequences containing 20 inversions are

5, 5, 4, 3, 2, 1, 0	6, 4, 4, 3, 2, 1, 0	6, 5, 3, 3, 2, 1, 0
6, 5, 4, 2, 2, 1, 0	6, 5, 4, 3, 1, 1, 0	6, 5, 4, 3, 2, 0, 0

[3] a.) have exactly 21 inversions? 1

The only inversion sequence containing 21 inversions is 6, 5, 4, 3, 2, 1, 0

[3] a.) have exactly 22 inversions? 0

[9] 4.) Draw the Hasse Diagram for the inversion poset $(X_3, \leq)$ where X_3 = the set of permutations of $\{1, 2, 3\}$ and if π and σ are two permutations in X_3, then $\pi \leq \sigma$ if the set of inversions of π is a subset of the set of inversions of σ.

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      321
 /          \
231          312
 |            |
213          132
 \           /
      123
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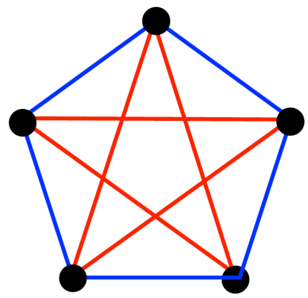
2pts Extra credit: Prove that $r(3,3) \geq 6$ (Note this problem is not the same as 6A).



The above coloring of K_5 shows there exists a coloring of K_5 that does not contain a red K_3 or a blue K_3. Thus $r(3,3) > 5$

[20] 5.) State the definition of equivalence relation:

An *equivalence relation* on X is a relation (i.e., a subset of $X \times X$) that is reflexive, symmetric, transitive.

Use the definition of equivalence relation to show the $\sim$ is an equivalence relation on $\mathbb{Z}$ where $n \sim k$ iff $\frac{n-k}{4} \in \mathbb{Z}$

Claim: $\cong_p$ is reflexive. That is, $\forall x \in X$, $x \sim x$.

$\frac{x-x}{4} = 0 \in \mathbb{Z}$. Thus $x \sim x$.

Claim: $\cong_p$ is symmetric. I.e., if $x \sim y$, then $y \sim x$.

Suppose $x \sim y$. Then $\frac{x-y}{4} \in \mathbb{Z}$. Thus $\frac{y-x}{4} = -\frac{x-y}{4} \in \mathbb{Z}$ since the negative of an integer is an integer. Hence $y \sim x$.

Claim: $\cong_p$ is transitive. I.e., if $x \sim y$ and $y \sim z$, then $x \sim z$.

Suppose $x \sim y$ and $y \sim z$. Then $\frac{x-y}{4} \in \mathbb{Z}$ and $\frac{y-z}{4} \in \mathbb{Z}$.

Thus $\frac{x-z}{4} = \frac{x-y}{4} + \frac{y-z}{4} \in \mathbb{Z}$ since the sum of two integers is an integer.

Thus $\sim$ is an equivalence relation.

What are the equivalence classes of $\mathbb{Z}$ with respect to $\sim$?

$[0] = \{..., -8, -4, 0, 4, 8, ...\}$ = the set containing all multiples of 4.

$[1] = \{..., -7, -3, 1, 5, 9, ...\}$ = the set of numbers whose remainder is 1 when divided by 4.

$[2] = \{..., -6, -2, 2, 6, 10, ...\}$ = the set of numbers whose remainder is 2 when divided by 4.

$[3] = \{..., -5, -1, 3, 7, 11, ...\}$ = the set of numbers whose remainder is 3 when divided by 4.

Partition $\mathbb{Z}$ into its equivalence classes (I.e., write $\mathbb{Z}$ as the disjoint union of sets where the sets correspond to equivalence classes.

$$\mathbb{Z} = [0] \cup [1] \cup [2] \cup [3]$$

[40] 6.) Choose 2 from the following 3 problems. **Circle your choices: A B C** You may do all 3 problems in which case your unchosen problem can replace your lowest scoring problem at 4/5 the value (or more) as discussed in class.

Note: If you do not CLEARLY indicate your 2 choices, I will assume that you chose the first two problems.

6A.) Prove that $r(3,3) \leq 6$. State where you use the Pigeon-hole principle and whether you use the weak form or the strong form.

Method 1: Proof by contradiction: Suppose $r(3,3) > 6$.

Then there exists a coloring of K_6 that contains neither a red triangle nor a blue triangle.

Let v be a vertex of K_6. Let us consider the set of red edges that have v as an endpoint and the set of blue edges that have v as an endpoint. The number of edges that have v as an endpoint is 5. Thus by the pigeon-hole principle (strong-form) of these 5 edges, either 3 are red or 3 are blue.

WLOG (without loss of generality) assume there exists 3 red edges, $\{v, v_1\}, \{v, v_2\}$, and $\{v, v_3\}$.

Consider the edges $\{v_1, v_2\}$, $\{v_1, v_3\}$, and $\{v_2, v_3\}$.

Suppose one of these edges, $\{v_i, v_j\}$ is colored red. Then $\{v, v_i\}$, $\{v, v_j\}$, $\{v_i, v_j\}$ is a red triangle. But our coloring does not have a red K_3.

Thus none of the edges $\{v_1, v_2\}$, $\{v_1, v_3\}$, and $\{v_2, v_3\}$ can be colored red.

Thus $\{v_1, v_2\}$, $\{v_1, v_3\}$, and $\{v_2, v_3\}$ are all colored blue. But then $\{v_1, v_2\}$, $\{v_1, v_3\}$, and $\{v_2, v_3\}$ is a blue triangle, a contridiction.

Thus there does not exist a coloring of K_6 that contains neither a red triangle nor a blue triangle.

Thus our assumption that $r(3,3) > 6$ is incorrect. Hend $r(3,3) \leq 6$.

If you are not required to use the pigeonhole principle, then two alternate proofs:

Method 2: $r(s,t) \leq r(s-1,t) + r(s,t-1)$ and $r(s,2) = r(2,s) = s$.

Thus $r(3,3) \leq r(2,3) + r(3,2) = 3 + 3 = 6$

Method 3: $r(s,t) \leq \binom{s+t-2}{s-1}$. Thus $r(3,3) \leq \binom{3+3-2}{3-1} = \binom{4}{2} = \frac{4!}{2!2!} = 6$.

6B.) Use a combinatorial argument to prove the Vandermonde convolution for the binomial coefficients: For all positve integers m_1, m_2, n,

$$\sum_{k=0}^{n} \binom{m_1}{k}\binom{m_2}{n-k} = \binom{m_1+m_2}{n}$$

See answers to HW ch 5 #25 posted under content on ICON

6C.) State Newton's binomial theorem for expanding $(x+y)^\alpha$ where $\alpha \in \mathbb{R}$.

Let $\alpha \in \mathcal{R}$. Then if $0 \leq |x| < |y|$,

$$(x+y)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k y^{\alpha-k}$$

where $\binom{\alpha}{k} = \begin{cases} \frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!} & \text{if } k \geq 1 \\ 1 & \text{if } k = 0 \\ 0 & \text{if } k \leq -1 \end{cases}$

Use this theorem to algebraically derive the formula: $\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k$ when $|z| < 1$.

Hint: Let $\alpha = -1$. You may use the fact that $\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}$

When $n = 1$: $\binom{-1}{k} = (-1)^k \binom{-1+k-1}{k} = (-1)^k \binom{k}{k} = (-1)^k$

When $|z| < 1$:

$$(x+y)^{-1} = \sum_{k=0}^{\infty} \binom{-1}{k} x^k y^{-1-k} = \sum_{k=0}^{\infty} (-1)^k x^k y^{-1-k}$$

$$\frac{1}{1-z} = (-z+1)^{-1} = \sum_{k=0}^{\infty} (-1)^k (-z)^k (1)^{-1-k} = \sum_{k=0}^{\infty} (-1)^k (-1)^k (z)^k = \sum_{k=0}^{\infty} z^k$$