Math 150 Exam 1 Form A
September 25, 2015

Choose 5 from the following 7 problems. Circle your choices: 1 2 3 4 5 6 7
You may do more than 6 problems in which case one of your two unchosen problems can replace your lowest problem at 4/5 the value (or more) as discussed in class.

1a.) The number of permutations of $n$ objects is $\underline{\ n!\ }$.

1b.) The number of ways to place $n$ non-attacking rooks on an $n \times n$ chessboard is $\underline{\ n!\ }$.

1c.) The number of $r-$permutations of $n$ objects is $\underline{\ \frac{n!}{(n-r)!}\ }$.

1d.) The number of ways to place $r$ nonattacking rooks on an $r \times n$ chessboard, $r \leq n$ is $\underline{\ \frac{n!}{(n-r)!}\ }$.

1e.) The number of ways to place $r$ nonattacking rooks on an $n \times n$ chessboard, $r \leq n$ is $\underline{\ C(n,r)P(n,r) = \frac{(n!)^2}{(r!)[(n-r)!]^2}\ }$.

1f.) $A = \{\infty \cdot 1, \infty \cdot 2, ..., \infty \cdot k\}$. The number of $r$ permutations of $A$ is $\underline{\ k^r\ }$.

1g.) Let $A = \{\infty \cdot 1, \infty \cdot 2, ..., \infty \cdot k\}$. The number of $r$ combinations of $A$ $\underline{\ \binom{r+k-1}{r}\ }$.

1h.) The number of subsets of $\{1, 2, ..., n\}$ containing exactly $r$ elements is $\underline{\ \binom{n}{r}\ }$.

1i.) The number of subsets of $\{1, 2, ..., n\}$ is $\underline{\ 2^n\ }$.

2a.) In how many ways can you place 12 identical rooks in non-attacking position on a $20 \times 20$ chessboard?

$C(20, 12)P(20, 12) = \frac{(20!)^2}{(12!)[8!]^2}$

2b.) In how many ways can you place 5 red rooks and 7 blue rooks in non-attacking position on a $20 \times 20$ chessboard?

$C(20, 12)P(20, 12)C(12, 5) = \frac{(20!)^2(12!)}{(12!)[8!]^2(5!)(7!)}$

2c) In how many ways can you place 5 red rooks and 7 blue rooks in non-attacking position on a $20 \times 20$ chessboard so that there is a rook in the 2nd and 4th columns?

Two colums already chosen, so just need to choose 2 more out of the remaining 18 columns:

$C(18, 10)P(20, 12)C(12, 5) = \frac{(18!)(20!)(12!)}{(10!)(8!)^2(5!)(7!)}$

3.) Suppose we have a group of 15 people that we label with the letters, A, B, C, D, E, F, G, H, I, J, K, L, M, N, O.

3a.) Find the number of arrangements of 10 of these 15 people seated around a circular table.

$$P(15,10)/10 = \frac{15!}{(5!)(10)}$$

3b.) What is the probability that A and B are among the 10 people chosen to sit at this table, but A does not sit next to B?

Number of ways for A and B chosen, but B does not sit next to A:

sit A = 1 (or 10 if you divide by 10 later).
sit B = 10 - 3 = 7
Choose 8 more people: $C(13,8)$
Sit these 8 people in the remaining 8 seats starting from A's right: 8!.
$\frac{(7)(8!)(13!)}{(8!)(5!)} = \frac{7(13!)}{5!}$

Alternatively:
sit A: 1
sit someone to the right of A (but not B): 15 - 2 = 13
sit someone to the left of A (but not B): 15 - 3 = 12
sit B in one of the remaining 7 seats: 7
sit someone in the remaining 6 seats: $\frac{11!}{(11-6)!}$
$\frac{(13)(12)(7)(11!)}{5!} = \frac{(7)(13!)}{5!}$

Probability $= \frac{\text{\# of ways for A \& B chosen, but B not next to A}}{\text{Number of arrangements of 10 of 15 people}} = \frac{(7)(13!)(5!)(10)}{5!(15!)} = \frac{1}{3}$

4.) Find the number of integral solutions to $y_1 + y_2 + y_3 = 21$ such that $y_1 \geq -1$, $y_2 \geq 0$, and $y_3 \geq 3$. Explain your answer including relating it to the number of permutations of $\{k \cdot 1, n \cdot +\}$.

Let $x_1 = y_1 + 1$. Let $x_2 = y_2$. $x_3 = y_3 - 3$.
$y_1 + y_2 + y_3 = 21$ is equivalent to $y_1 + 1 + y_2 + y_3 - 3 = 21 + 1 - 3 = 19$
$x_1 + x_2 + x_3 = 19$
the number of integral solutions to $x_1 + x_2 + x_3 = 19$ = permutations of $\{19 \cdot 1, 2 \cdot +\} = \frac{21!}{(19!)(2!)}$

A solution to $x_1 + x_2 + x_3 = 19$ can be uniquely represented by a permutation of $\{19 \cdot 1, 2 \cdot +\}$.
The solution to $x_1$ is the number of 1's before the first +.
The solution to $x_2$ is the number of 1's between the two +'s.
The solution to $x_3$ is the number of 1's after the last +.
For example, the solution $x_1 = 10, x_2 = 3, x_3 = 6$ is represented by the permutation

1111111111+111+111111

5.) Circle $T$ for true and $F$ for false. **If the statement is false, prove it by giving a counter-example.**

5a.) The function $f : R \to R$, $f(x) = x^2 + 3$ is onto F

$x^2 + 3 \geq 3$. Thus $x^2 + 3 = 0$ has no solution and thus $0 \in R$ is not in the image of $f$ and thus $f$ is not onto.

5b.) The function $f : R \to R$, $f(x) = x^2 + 3$ is 1:1. F

$f(1) = (1)^2 + 3 = 4 = (-1)^2 + 3 = f(-1)$. Thus $f$ is not 1:1.

5c.) If the statement $p$ implies $q$ is true, then its contrapositive, $\sim q$ implies $\sim p$, is also true. T

5d.) If the statement $p$ implies $q$ is true, then its converse, $q$ implies $p$, is also true. F

$x = 1$ implies $x > 0$. But $x > 0$ does not imply $x = 1$.

6.) Prove that for any $n + 1$ integers, $a_1, a_2, ..., a_{n+1}$, there exist two of the integers $a_i, a_j$ with $i \neq j$ such that $a_i - a_j$ is divisible by $n$.

$n + 1$ objects: $a_1, a_2, ..., a_{n+1}$

$n$ Boxes = remainder = $\{0, 1, ..., n - 1\}$

Proof: The set $\{a_1, a_2, ..., a_{n+1}\}$ has $n + 1$ objects. The set of possible remainders when divided by $n$ is $\{0, 1, ..., n - 1\}$. Thus there are $n$ possible remainders. Thus by the pigeon hole principle (weak form), there exists $i \neq j$ such that $a_i$ and $a_j$ have the same remainder when divided by $n$. Thus there exists integers $d_1$ and $d_2$ such that $a_i = nd_1 + r$ and $a_j = nd_2 + r$. Thus $a_i - a_j = nd_1 + r - (nd_2 + r) = n(d_1 - d_2)$. Thus $a_i - a_j$ is divisible by $n$.

7) Let $B = \{n_1 \cdot 1, n_2 \cdot 2, ..., n_k \cdot k\}$. Let $n = n_1 + n_2 + ... + n_k$. Prove that the number of permutations of $B = \frac{n!}{n_1! n_2! \cdots n_k!}$.

See book (Thm 2.4.2) or class notes.