$\sqrt{1+4} \neq \sqrt{1}+\sqrt{4}$ $\qquad$ $\frac{1}{2+3} \neq \frac{1}{2}+\frac{1}{3}$

not linear

Linear Functions

A function f is linear if $f(a\mathbf{x}+b\mathbf{y}) = af(\mathbf{x})+bf(\mathbf{y})$

Or equivalently f is linear if
1.) $f(a\mathbf{x}) = af(\mathbf{x})$ and 2.) $f(\mathbf{x}+\mathbf{y}) = f(\mathbf{x})+f(\mathbf{y})$

Theorem: If f is linear, then $f(\mathbf{0}) = \mathbf{0}$

Proof: $f(\mathbf{0}) = f(0\cdot\mathbf{0}) = 0\cdot f(\mathbf{0}) = \mathbf{0}$

Example 1.) $f: R \to R$, $f(x) = 2x$

Proof:
$f(ax+by) = 2(ax+by) = 2ax+2by = af(x)+bf(y)$

Example 2.) $f: R^2 \to R^2$,
$f((x_1,x_2)) = (2x_1, x_1+x_2)$

Proof: Let $\mathbf{x} = (x_1,x_2)$, $\mathbf{y} = (y_1,y_2)$

$a\mathbf{x}+b\mathbf{y} = a(x_1,x_2)+b(y_1,y_2) = (ax_1,ax_2)+(by_1,by_2) = (ax_1+by_1, ax_2+by_2)$

$f(ax_1+by_1, ax_2+by_2)$

$$= (2(ax_1+by_1), ax_1+by_1+ax_2+by_2)$$

$$= (2ax_1+2by_1, ax_1+ax_2+by_1+by_2)$$

$$= (2ax_1, ax_1+ax_2)+(2by_1, by_1+by_2)$$

$$= a(2x_1, x_1+x_2)+b(2y_1, y_1+y_2)$$

$$= af((x_1,x_2))+bf((y_1,y_2))$$

Example 3.) D : set of all differential functions $\to$ set of all functions, $D(f) = f'$

Proof:
$D(af+bg) = (af+bg)' = af'+bg' = aD(f)+bD(g)$

Example 4.) Given a, b real numbers,
I : set of all integrable functions on [a, b] $\to$ R,
$I(f) = \int_a^b f$

Proof: $I(sf+tg) = \int_a^b sf+tg = s\int_a^b f + t\int_a^b g = sI(f)+tI(g)$

Example 5.) The inverse of a linear function is linear (when the inverse exists).

Suppose $f^{-1}(x) = c$, $f^{-1}(y) = d$.

Then $f(c) = x$ and $f(d) = y$ and
$f(ac + bd) = af(c) + bf(d) = ax + by$.

Hence $f^{-1}(ax + by) = ac + bd = af^{-1}(x) + bf^{-1}(y)$.

Example 6.) D : set of all twice differential functions $\to$ set of all functions, $L(f) = af'' + bf' + cf$

Proof:

$$L(sf + tg) = a(sf + tg)'' + b(sf + tg)' + c(sf + tg)$$

$$= saf'' + tag'' + sbf' + tbg' + scf + tcg$$

$$= s(af'' + bf' + cf) + t(ag'' + bg' + cg)$$

$$= sL(f) + tL(g)$$

Suppose $L(\psi_1) = 0$ $\quad L(\psi_2) = 0$
and $L(\varphi) = b$

$$L(c_1\psi_1 + c_2\psi_2 + \varphi) = c_1 L(\psi_1) + c_2 L(\psi_2) + L(\varphi)$$

Consequence 1: If ψ_1, ψ_2 are solutions to $af'' + bf' + cf = 0$, then $3\psi_1 + 5\psi_2$ is also a solution to $af'' + bf' + cf = 0$,

Proof: Since ψ_1, ψ_2 are solutions to $af'' + bf' + cf = 0$
$L(\psi_1) = 0$ and $L(\psi_2) = 0$.

Hence $L(3\psi_1 + 5\psi_2) = 3L(\psi_1) + 5L(\psi_2)$
$= 3(0) + 5(0) = 0.$

Thus $3\psi_1 + 5\psi_2$ is also a solution to $af'' + bf' + cf = 0$

Consequence 2:
If ψ_1 is a solution to $af'' + bf' + cf = h$
and ψ_2 is a solution to $af'' + bf' + cf = k$,
then $3\psi_1 + 5\psi_2$ is a solution to $af'' + bf' + cf = 3h + 5k$,

Since ψ_1 is a solution to $af'' + bf' + cf = h$, $L(\psi_1) = h$.

Since ψ_2 is a solution to $af'' + bf' + cf = k$, $L(\psi_2) = k$.

Hence $L(3\psi_1 + 5\psi_2) = 3L(\psi_1) + 5L(\psi_2)$
$= 3h + 5k.$

Thus $3\psi_1 + 5\psi_2$ is also a solution to
$af'' + bf' + cf = 3h + 5k$

$L(h_n) = h_n - a_1 h_{n-1} - \dots a_k h_{n-k}$

seq

Claim L is a linear function

7.4: linear homogeneous recurrence relation:

$$h_n - a_1 h_{n-1} - a_2 h_{n-2} - \dots - a_k h_{n-k} = 0$$

plug it in

Suppose $\phi(n)$ and $\psi(n)$ are solns to the above recurrence relation, then

$$\phi_n - a_1 \phi_{n-1} - a_2 \phi_{n-2} - \dots - a_k \phi_{n-k} = 0$$

$$\psi_n - a_1 \psi_{n-1} - a_2 \psi_{n-2} - \dots - a_{n-k} \psi_{n-k} = 0$$

Claim 1: $c\phi(n)$ is a solution for any constant c

$$c\phi_n - a_1 c\phi_{n-1} - a_2 c\phi_{n-2} - \dots - a_k c\phi_{n-k}$$

$$= c(\phi_n - a_1 \phi_{n-1} - a_2 \phi_{n-2} - \dots - a_k \phi_{n-k})$$

$$= c(0) = 0 \quad \square$$

Claim 2: $\phi(n) + \psi(n)$ is also a solution.

Let $h_n = \phi_n + \psi_n$

$$\phi_n + \psi_n - a_1(\phi_{n-1} + \psi_{n-1}) - a_2(\phi_{n-2} + \psi_{n-2}) - \dots - a_k(\phi_{n-k} + \psi_{n-k})$$

$$= \phi_n - a_1 \phi_{n-1} - a_2 \phi_{n-2} - \dots - a_k \phi_{n-k} + \psi_n - a_1 \psi_{n-1} - a_2 \psi_{n-2} - \dots - a_k \psi_{n-k}$$

$$= 0 + 0 = 0$$

Hence if $\phi_i(n)$ are solns, then $\Sigma c_i \phi_i(n)$ is a soln for any constants c_i.

any linear combination is also a sol'n (ignoring initial conditions)

Thm 7.4.1: Suppose a_i are constants and $q \neq 0$. Then q^n is a solution to

$$\boxed{h_n - a_1 h_{n-1} - a_2 h_{n-2} - \dots - a_k h_{n-k} = 0}$$

same equation when $h_n = q^n$

iff q is a root of the polynomial equation

$$x^k - a_1 x^{k-1} - a_2 x^{k-2} - \dots - a_k = 0$$

If this characteristic equation has k distinct roots, $q_1, q_2, \dots, q_k$, then $h_n = c_1 q_1^n + c_2 q_2^n + \dots + c_k q_k^n$ is the general solution.

linear comb of homog sol'ns is a homog sol'n

I.e. given any initial values for $h_0, h_1, \dots, h_{k-1}$, there exists $c_1, c_2, \dots, c_k$ such that $h_n = c_1 q_1^n + c_2 q_2^n + \dots + c_k q_k^n$ satisfies the recurrence relation and the initial conditions.

Thm 7.4.2: Suppose q_i is an s_i-fold root of the characteristic equation. Then

$$H_i(n) = c_1 q_i^n + c_2 n q_i^n + \dots + c_{s_i} n^{s_i - 1} q_i^n$$

is a solution to the recurrence relation.

If the characteristic equation has t distinct roots $q_1, \dots, q_t$ with multiplicity $s_1, \dots, s_t$, respectively, then

$h_n = H_1(n) + \dots H_t(n)$ is a general solution.

homog

7.4: linear homogeneous recurrence relation:

Defn: A recurrence relation is *linear* if

$$h_n = a_1(n)h_{n-1} + a_2(n)h_{n-2} + \dots + a_k(n)h_{n-k} + b(n)$$

A recurrence relation has order k if $a_k \neq 0$

Ex: Derangement

$$D_n = (n-1)D_{n-1} + (n-1)D_{n-2}, \qquad D_1 = 0, \quad D_2 = 1$$

order 2

nonhomog

$$D_n = nD_{n-1} + (-1)^n, \qquad D_1 = 0$$

order 1

Fibonacci: $f_n = f_{n-1} + f_{n-2}, \qquad f(0) = 0, \ f(1) = 1$

order 2

Defn: A linear recurrence relation is *homogeneous* if $b = 0$

$h_n - a_1 h_{n-1} - a_2 h_{n-2} \cdots - a_k h_{n-k} = b$

Defn: A linear recurrence relation has *constant coefficients* if the a_i's are constant.

Tentative HW 12: Ch 14: 1, 4, 5, 10, 13, 18, 22, 24, 25 and

A.) Suppose the sequences r_n, s_n, and t_n satisfy the homogeneous linear recurrence relation,

$$h_n = a_1(n)h_{n-1} + a_2(n)h_{n-2} + a_3(n)h_{n-3} \ (**).$$

Show that the sequence, $c_1r_n + c_2s_n + c_3t_n$ also satisfies this homogeneous linear recurrence relation (**).

B.) Suppose the sequence ψ_n satisfies the linear recurrence reln, $h_n = a_1(n)h_{n-1} + a_2(n)h_{n-2} + a_3(n)h_{n-3} + b(n)$ (*). Show that the sequence, $c_1r_n + c_2s_n + c_3t_n + \psi_n$ also satisfies this linear recurrence relation.

C.) How many terms of the sequence are needed to find a unique sequence with these terms satisfying (*). What linear system of equations can be used to determine c_1, c_2, c_3.

7.4: linear homogeneous recurrence relation w/constant coefficients:

Ex: Solve the recurrence relation: $h_n + h_{n-2} = 0$, $h_0 = 3, h_1 = 5$

Guess q^n is a solution.

$q^n + q^{n-2} = q^{n-2}(q^2+1) = 0 \qquad\qquad q^2 + 1 = 0$ implies $q = \pm i$

Thus the general solution is $h_n = c_1 i^n + c_2(-i)^n$

i.e., this function satisfies the recurrence relation.

linear comb

Now need to find c_i's resulting in initial conditions:

$h_0 = 3$: $c_1 + c_2 = 3$ implies $c_2 = 3 - c_1$

$h_1 = 5$: $c_1 i - c_2 i = 5$ implies $-c_1 + c_2 = 5i$

$-c_1 + 3 - c_1 = 5i$. Thus $-2c_1 + 3 = 5i$

Hence $c_1 = \frac{3-5i}{2}$ and $c_2 = 3 - (\frac{3-5i}{2}) = \frac{3+5i}{2}$

plug in initial conditions to find c_1 & c_2

$h_n = (\frac{3-5i}{2})i^n + (\frac{3+5i}{2})(-i)^n$ satisfies the recurrence relation and the initial conditions.

$h_n = i^n[(\frac{3-5i}{2}) + (\frac{3+5i}{2})(-1)^n] = i^n[(\frac{3}{2})(1 + (-1)^n) + (\frac{5i}{2})(-1 + (-1)^n)]$

$h_{2j} = (\frac{3-5i}{2})i^{2j} + (\frac{3+5i}{2})(-i)^{2j} = 3(-1)^j$

$h_{2j+1} = (\frac{3-5i}{2})i^{2j+1} + (\frac{3+5i}{2})(-i)^{2j+1} = -5(i)^{2j+2} = 5(-1)^j$

Thus starting with h_0, we have the sequence:

3, 5, -3, -5, 3, 5, -3, -5, 3, 5, -3, -5, 3, 5, ...

1, 1, 1, 1, 1, 1, ...

← need 4 linearly independent sol'ns

Ex: Solve the recurrence relation, $h_n - 2h_{n-1} + 2h_{n-3} - h_{n-4} = 0$, $h_0 = 3, h_1 = 3, h_2 = 7, h_3 = 15$.

Guess q^n is a solution.

$q^n - 2q^{n-1} + 2q^{n-3} - q^{n-4} = q^{n-4}(q^4 - 2q^3 + 2q - 1) = 0$,

$q^{n-4}(q^3 - 3q^2 + 3q - 1)(q+1) = q^{n-4}(q-1)^3(q+1) = 0$

q = 1 , 1, 1, -1

Note: **1 is a repeated root**

Note $n^j(1)^n$, $j = 0, 1, 2$, are solutions to the recurrence relation.

Check: If $h_n = (1)^n = 1$: $1 - 2 + 2 - 1 = 0$.

Check: If $h_n = n(1)^n = n$: 1, 2, 3, 4, 5, ...

$n - 2(n-1) + 2(n-3) - (n-4) = n - 2n + 2n - n + 2 - 6 + 4 = 0$

Check: If $h_n = n^2(1)^n = n^2$: 1, 4, 9, 16, ...

$n^2 - 2(n-1)^2 + 2(n-3)^2 - (n-4)^2 =$
$n^2 - 2(n^2 - 2n + 1) + 2(n^2 - 6n + 9) - (n^2 - 8n + 16) = 0$

4 l.i. sol'ns

General solution

$h_n = c_1(1)^n + c_2 n(1)^n + c_3 n^2(1)^n + c_4(-1)^n = c_1 + c_2 n + c_3 n^2 + c_4(-1)^n$

Now need to find c_i's resulting in initial conditions:

$h_0 = 3 = c_1 + c_4$

$h_1 = 3 = c_1 + c_2 + c_3 - c_4$

$h_2 = 7 = c_1 + 2c_2 + 4c_3 + c_4$

$h_3 = 15 = c_1 + 3c_2 + 9c_3 - c_4$

$h_0 = 3 = c_1 + c_4$

$h_1 = 3 = c_1 + c_2 + c_3 - c_4$

$h_2 = 7 = c_1 + 2c_2 + 4c_3 + c_4$

$h_3 = 15 = c_1 + 3c_2 + 9c_3 - c_4$

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 3 \\ 1 & 1 & 1 & -1 & 3 \\ 1 & 2 & 4 & 1 & 7 \\ 1 & 3 & 9 & -1 & 15 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 2 & 4 & 0 & 4 \\ 0 & 3 & 9 & -2 & 12 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 2 & 4 & 4 \\ 0 & 0 & 6 & 4 & 12 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 2 & 4 & 4 \\ 0 & 0 & 0 & -8 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 2 & 4 & 4 \\ 0 & 0 & 0 & -8 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Thus $c_1 = 3$, $c_2 = -2$, $c_3 = 2$, $c_4 = 0$.

$h_n = c_1 + c_2 n + c_3 n^2 + c_4(-1)^n = 3 - 2n + 2n^2$

Hence $\boxed{h_n = 3 - 2n + 2n^2}$ ← Answer

Check Initial Conditions: $h_0 = 3, h_1 = 3, h_2 = 7, h_3 = 15$

$h_0 = 3 - 0 + 0 = 3$ $\qquad$ $h_1 = 3 - 2 + 2 = 3$,
$h_2 = 3 - 4 + 8 = 7$ $\qquad$ $h_3 = 3 - 6 + 18 = 15$.

3, 3, 7, 15, $3 - 2(4) + 2(4)^2$, ...