Thus $f_n = 0$, $f_n = \left(\frac{1+\sqrt{5}}{2}\right)^n$ and $f_n = \left(\frac{1-\sqrt{5}}{2}\right)^n$

are 3 different sequences that satisfy the

homogeneous linear recurrence relation: $f_n - f_{n-1} - f_{n-2} = 0$.

Hence $f_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$ also satisfies the

homogeneous linear recurrence relation: $f_n - f_{n-1} - f_{n-2} = 0$.

Suppose the initial conditions are $f_0 = a$ and $f_1 = b$

(note for fibonacci sequence, $a = 0$ and $b = 1$).

Then for $n = 0$: $f_0 = c_1 + c_2 = a$

And for $n = 1$: $f_1 = c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right) = b$

Or in matrix form: $\begin{pmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \frac{1-\sqrt{5}}{-2\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1+\sqrt{5}}{2\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{1-\sqrt{5}}{-2\sqrt{5}}a + \frac{b}{\sqrt{5}} \\ \frac{1+\sqrt{5}}{2\sqrt{5}}a - \frac{b}{\sqrt{5}} \end{pmatrix}$$

If $a = 0$ and $b = 1$, then $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{pmatrix}$

5.6

$\frac{1}{(1-x)^n} = \Sigma_{k=0}^{\infty}\binom{-n}{k}(-x)^k = \Sigma_{k=0}^{\infty}\binom{n+k-1}{k}x^k$, $|x| < 1$

$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$ geometric series

$\frac{x^{n+1}-1}{x-1} = \underbrace{1 + x + x^2 + x^3 + \ldots + x^n}_{\text{partial sums of geometric}}$

x^0 x^1 x^2 x^3

0th 1rst 2nd 3rd

7.2: Generating Functions

$g(x) = h_0 + h_1x + h_2x^2 + \ldots$ is the *generating function* for the sequence $h_0, h_1, h_2, \ldots$

Ex: The generating fn for the sequence 2, 3, 4, 0, 0, 0, ... is

$$g(x) = 2 + 3x + 4x^2$$

Ex: The generating function for the sequence 1, 1, 1, ... is

$$g(x) = 1 + x + x^2 + \ldots = \frac{1}{1-x}$$

$|x| < 1$

geometric series

Ex: The generating function for the sequence
0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, ... is

$$g(x) = x^4 + x^7 + x^{10} + \ldots = x^4(1 + x^3 + x^6 + \ldots) = \frac{x^4}{1-x^3}$$

geometric series

x^3

Ex: The generating function for the sequence

$\binom{m}{0}, \binom{m}{1}, \binom{m}{2}, \ldots, \binom{m}{m}$ is $\qquad |x|<1$

$$\boxed{g(x) = \binom{m}{0} + \binom{m}{1}x + \binom{m}{2}x^2 + \ldots \binom{m}{m}x^m = (1+x)^m}$$

Ex: Suppose $\alpha \in \mathcal{R}$. The generating function for the sequence

$\binom{\alpha}{0}, \binom{\alpha}{1}, \binom{\alpha}{2}, \ldots$ is $\qquad |x|<1$

$$\boxed{g(x) = \binom{\alpha}{0} + \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 + \ldots = (1+x)^\alpha}$$

Ex: Let h_n = number of nonnegative solutions to

$$e_1 + e_2 + \ldots + e_k = n$$

Thus $h_n = \binom{n+k-1}{n}$

$|x|<1$

Thus $g(x) = \sum_{n=0}^{\infty} \underbrace{\binom{n+k-1}{n}} X^n = \frac{1}{(1-X)^k}$

switched k & n in formula on preceding page

7.2 $\{\infty \cdot x, \infty \cdot y\}$ multisets

list of all possible sets containing elements satisfying

Suppose a multiset of size k must contain the following:

- between two to four (inclusive) x's,
- zero, one, two or five y's.

Find the number of multisets of size k.

"Long" method: list all possibilities

between two to four (inclusive) x's: $\quad x^2 + x^3 + x^4$

zero, one, two or five y's: $\quad y^0 + y^1 + y^2 + y^5$

Both: $(x^2 + x^3 + x^4)(y^0 + y^1 + y^2 + y^5)$

$$\cancel{= x^2y^0 + x^2y^1 + x^2y^2 + x^2y^5 + x^3y^0 + x^3y^1 + x^3y^2 + x^3y^5}$$
$$\cancel{+ x^4y^0 + x^4y^1 + x^4y^2 + x^4y^5}$$

$$= x^2y^0 + (x^2y^1 + x^3y^0) + (x^2y^2 + x^3y^1 + x^4y^0)$$
$$+ (x^3y^2 + x^4y^1) + x^4y^2 + x^2y^5 + x^3y^5 + x^4y^5$$

$y \to x$

Let h_k = number of multisets of size k.

$h_0 = 0$, $h_1 = 0$, $h_2 = 1$, $h_3 = 2$, $h_4 = 3$,
$h_5 = 5$, $h_6 = 1$, $h_7 = 1$, $h_8 = 1$, $h_9 = 1$,
$h_k = 0$ $\quad k > 9$

"Shorter" method:

between two to four (inclusive) x's: $\quad x^2 + x^3 + x^4$

zero, one, two or five y's: $\quad x^0 + x^1 + x^2 + x^5$

Both: $\quad g(x) = (x^2 + x^3 + x^4)(x^0 + x^1 + x^2 + x^5)$

$$= x^2x^0 + (x^2x^1 + x^3x^0) + (x^2x^2 + x^3x^1 + x^4x^0)$$
$$+ (x^3x^2 + x^4x^1) + x^4x^2 + x^2x^5 + x^3x^5 + x^4x^5$$

$$= x^2 + 2x^3 + 3x^4 + 2x^5 + x^6 + x^7 + x^8 + x^9$$

$h_0 = h_1 = 0$

$h_2 = 1 \quad h_3 = 2 \quad h_4 = 3, \ldots$