Note most quizzes will be 5 minutes in length and are meant to cover basic material. Thus if you are not averaging an A on quizzes, you are probably failing this class. This is really true. Past classes bear strong witness to this fact.

Thus for each quiz, make sure you study the basics as outlined in each quiz study guide (click on SG for the appropriate quiz for the Study guide.

You will have more time on exams, so exam questions can be more challenging (so practice more than just the basics).

Quiz 5 study guide (note quizzes are cumulative):

Find the generating function of a sequence and simplify (See Monday's 11/16 lecture).

$h_0, h_1, h_2, \ldots$

$h_0 + h_1 x + h_2 x^2$

Define Fibonacci sequence via (1) give first several terms of sequence, (2) by stating the homogeneous linear recurrence relation plus initital conditions.

$0, 1, 1, 2, 3, 5, 8, 13, \ldots$

$$\boxed{\begin{array}{l} f_n - f_{n-1} - f_{n-2} = 0 \\ f_0 = 0, \; f_1 = 1 \end{array}}$$

Define geometric sequence.

State the formula for the partial sums of a geometric sequence.

Define derangement.

Calculate D_n.

State the inclusion-exclusion theorem.

Problem similar to: How many terms are in the sum $\sum |A_i \cap A_j \cap A_k|$? $= \binom{n}{3}$ if there are n A_ℓ's

State Pascal's formula.

Define inversion, disorder, even permutation, odd permutation.

Thus $f_n = 0$, $f_n = \left(\frac{1+\sqrt{5}}{2}\right)^n$ and $f_n = \left(\frac{1-\sqrt{5}}{2}\right)^n$

are 3 different sequences that satisfy the

homogeneous linear recurrence relation: $f_n - f_{n-1} - f_{n-2} = 0$.

Hence $f_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$ also satisfies the

homogeneous linear recurrence relation: $f_n - f_{n-1} - f_{n-2} = 0$.

Suppose the initial conditions are $f_0 = a$ and $f_1 = b$

(note for fibonacci sequence, $a = 0$ and $b = 1$).

Then for $n = 0$: $f_0 = c_1 + c_2 = a$

And for $n = 1$: $f_1 = c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right) = b$

Or in matrix form: $\begin{pmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \frac{1-\sqrt{5}}{-2\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1+\sqrt{5}}{2\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{1-\sqrt{5}}{-2\sqrt{5}}a + \frac{b}{\sqrt{5}} \\ \frac{1+\sqrt{5}}{2\sqrt{5}}a - \frac{b}{\sqrt{5}} \end{pmatrix}$$

If $a = 0$ and $b = 1$, then $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{pmatrix}$

5.6

$\frac{1}{(1-x)^n} = \Sigma_{k=0}^{\infty}\binom{-n}{k}(-x)^k = \Sigma_{k=0}^{\infty}\binom{n+k-1}{k}x^k$, $|x| < 1$

$\boxed{\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots}$ geometric series

x^0 x^1 x^2 x^3

$\boxed{\frac{x^{n+1}-1}{x-1} = \underbrace{1 + x + x^2 + x^3 + \dots + x^n}_{\text{partial sums of geometric}}}$

0th 1rst 2nd 3rd

7.2: Generating Functions

0 1 2

$g(x) = h_0 + h_1x + h_2x^2 + \dots$. is the *generating function* for the sequence $h_0, h_1, h_2, \dots$..

Ex: The generating fn for the sequence 2, 3, 4, 0, 0, 0, ... is

$$g(x) = 2 + 3x + 4x^2$$

Ex: The generating function for the sequence 1, 1, 1, ... is

$$g(x) = 1 + x + x^2 + \dots = \frac{1}{1-x}$$ $|x| < 1$

geometric series

Ex: The generating function for the sequence
0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, ... is

$$g(x) = x^4 + x^7 + x^{10} + \dots = x^4(1 + x^3 + x^6 + \dots) = \frac{x^4}{1-x^3}$$

geometric series

x^3

Ex: The generating function for the sequence

$\binom{m}{0}, \binom{m}{1}, \binom{m}{2}, \ldots, \binom{m}{m}$ is

$$g(x) = \binom{m}{0} + \binom{m}{1} x + \binom{m}{2} x^2 + \ldots \binom{m}{m} x^m = (1+x)^m$$

Ex: Suppose $\alpha \in \mathcal{R}$. The generating function for the sequence

$\binom{\alpha}{0}, \binom{\alpha}{1}, \binom{\alpha}{2}, \ldots$ is

$$g(x) = \binom{\alpha}{0} + \binom{\alpha}{1} x + \binom{\alpha}{2} x^2 + \ldots = (1+x)^\alpha$$

Ex: Let h_n = number of nonnegative solutions to

$$e_1 + e_2 + \ldots + e_k = n$$

Thus $h_n =$

Thus $g(x) =$

Suppose a multiset of size k must contain the following:

between two to four (inclusive) x's,
zero, one, two or five y's.

Find the number of multisets of size k.

"Long" method: list all possibilities

between two to four (inclusive) x's: $x^2 + x^3 + x^4$

zero, one, two or five y's: $y^0 + y^1 + y^2 + y^5$

Both: $(x^2 + x^3 + x^4)(y^0 + y^1 + y^2 + y^5)$

$$= x^2y^0 + x^2y^1 + x^2y^2 + x^2y^5 + x^3y^0 + x^3y^1 + x^3y^2 + x^3y^5 + x^4y^0 + x^4y^1 + x^4y^2 + x^4y^5$$

$$= x^2y^0 + (x^2y^1 + x^3y^0) + (x^2y^2 + x^3y^1 + x^4y^0) + (x^3y^2 + x^4y^1) + x^4y^2 + x^2y^5 + x^3y^5 + x^4y^5$$

Let h_k = number of multisets of size k.

$h_0 =$, $h_1 =$, $h_2 =$, $h_3 =$, $h_4 =$,
$h_5 =$, $h_6 =$, $h_7 =$, $h_8 =$, $h_9 =$,
$h_k =$ $k > 9$

"Shorter" method:

between two to four (inclusive) x's: $x^2 + x^3 + x^4$

zero, one, two or five y's: $x^0 + x^1 + x^2 + x^5$

Both: $g(x) = (x^2 + x^3 + x^4)(x^0 + x^1 + x^2 + x^5)$

$$= x^2x^0 + (x^2x^1 + x^3x^0) + (x^2x^2 + x^3x^1 + x^4x^0) + (x^3x^2 + x^4x^1) + x^4x^2 + x^2x^5 + x^3x^5 + x^4x^5$$

$$= x^2 + 2x^3 + 3x^4 + 2x^5 + x^6 + x^7 + x^8 + x^9$$

$g_0 = \sum_{k=0}^{-1} \binom{?}{} = 0 = f_0$

Thm 7.1.2: $f_n = \sum_{k=0}^{n-1} \binom{n-1-k}{k}$

Proof: Check if $\boxed{g(n) = \sum_{k=0}^{n-1} \binom{n-1-k}{k}}$

satisfies $g(n) = g(n-1) + g(n-2)$ and $g(1) = 1$ and $g(2) = 1$

$g(1) = \sum_{k=0}^{1-1} \binom{1-1-k}{k} = \sum_{k=0}^{0} \binom{-k}{k} = \binom{0}{0} = 1$ ✓

$g(2) = \sum_{k=0}^{2-1} \binom{2-1-k}{k} = \sum_{k=0}^{1} \binom{1-k}{k} = \binom{1}{0} + \binom{0}{1} = 1 + 0 = 1$ ✓

initial conditions hold

$g(n-1) + g(n-2)$ ← check recurrence rel'n

$= \sum_{k=0}^{(n-1)-1} \binom{(n-1)-1-k}{k} + \sum_{k=0}^{(n-2)-1} \binom{(n-2)-1-k}{k}$

$= \sum_{k=0}^{n-2} \binom{n-2-k}{k} + \sum_{k=0}^{n-3} \binom{n-3-k}{k}$

$= \sum_{k=0}^{n-2} \binom{n-2-k}{k} + \sum_{k=1}^{n-2} \binom{n-3-(k-1)}{k-1}$

$= \binom{n-2}{0} + \sum_{k=1}^{n-2} \binom{n-2-k}{k} + \sum_{k=1}^{n-2} \binom{n-3-(k-1)}{k-1}$

$= \binom{n-2}{0} + \sum_{k=1}^{n-2} \binom{n-2-k}{k} + \sum_{k=1}^{n-2} \binom{n-2-k}{k-1}$

$= \binom{n-2}{0} + \sum_{k=1}^{n-2} \left[\binom{n-2-k}{k} + \binom{n-2-k}{k-1} \right]$

$= \binom{n-2}{0} + \sum_{k=1}^{n-2} \binom{n-1-k}{k}$

$= \binom{n-2}{0} + \sum_{k=0}^{n-1} \binom{n-1-k}{k} - \binom{n-1}{0} - \binom{0}{n-1}$

$= 1 + \sum_{k=0}^{n-1} \binom{n-1-k}{k} - 1 - 0$

$= \sum_{k=0}^{n-1} \binom{n-1-k}{k} = g(n)$

$f_n = f_{n-1} + f_{n-2}$

Fibonacci sequence is defined by

Homogeneous linear recurrence relation: $f_n - f_{n-1} - f_{n-2} = 0$

and initial conditions: $f(0) = 0$, $f(1) = 1$.

Thm 7.1.1: $f_n = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n$

Proof: Suppose $f_n = x^n$. Then $f_{n-1} = x^{n-1}$ and $f_{n-2} = x^{n-2}$

Then $0 = f_n - f_{n-1} - f_{n-2} = x^n - x^{n-1} - x^{n-2}$

Thus $x^{n-2}(x^2 - x - 1) = 0$.

Thus either $x = 0$ or $x = \frac{1 \pm \sqrt{1-4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$

$h_{100} = 3 + 100(5)$

7.1: Sequences

Arithmetic sequence: $h_0, h_0+q, h_0+2q, \ldots$

$h_n = h_{n-1} + q = h_0 + nq,\ n \geq 0$

h_0

Example: $h_n = 3+5n$: 3, 8, 13, 18, 23, 28, ...

Geometric sequence: $h_0, qh_0, q^2h_0, \ldots$

$h_n = qh_{n-1} = q^n h_0,\ n \geq 0$

Example: $h_n = 2^n$: $1, 2, 4, 8, 16, 32, 62, 128, 256, 512, \ldots$

$h_n = 2^n$ = number of combinations of an n-element set.

Partial sums: $s_n = \sum_{k=0}^{n} h_k$

Partial sums of arithmetic sequence:

$$s_n = \sum_{k=0}^{n} h_0 + kq = \sum_{k=0}^{n} h_0 + \sum_{k=0}^{n} kq = (n+1)h_0 + \frac{qn(n+1)}{2}$$

Example: If $h_k = 3+5k$, then $s_n = \sum_{k=0}^{n} h_k = (n+1)3 + \frac{5n(n+1)}{2}$

3, 11, 24, 42, 65, 93, ...

$$\boxed{\text{Geometric sequence: } s_n = \sum_{k=0}^{n} q^k h_0 = \begin{cases} \frac{q^{n+1}-1}{q-1} h_0 & q \neq 1 \\ (n+1)h_0 & q = 1 \end{cases}}$$

Example: If $h_k = 2^k$, then $s_n = \sum_{k=0}^{n} h_k = \frac{2^{n+1}-1}{2-1}$

1, 3, 7, 15, 31, 63, ...

Fibonacci:

$$\boxed{\begin{array}{l} f_0 = 0,\ f_1 = \\ f_n = f_{n-1} + f_{n-2} \end{array}}$$

Suppose a pair of rabbits of the opposite sex give birth to a pair of rabbits of opposite sex every month starting with their second month. If we begin with a pair of newly born rabbits, how many rabbits are there after one year.

Let f_n = # of pairs of rabbits at the beginning of month n

$f_0 = 0 \quad f_1 = 1 \quad f_2 = 1 \quad f_3 = 2 \quad f_4 = 3 \quad f_5 = 3+2$

Hence $f_n = f_{n-1} + f_{n-2}$ 2 months have passed

Lemma: $s_n = \sum_{k=0}^{n} f_n = f_{n-2} - 1$

Proof by induction on n.

Lemma: f_n is even iff $3|n$.

Proof by induction on n.

Note that $f_0 = 0$ is even, $f_1 = 1$ is odd, and $f_2 = 1$ is odd. base case

Suppose f_{3n} is even, f_{3n+1} is odd, and f_{3n+2} is odd.

Then $f_{3n+3} = f_{3n+2} + f_{3n+1}$. Since odd + odd is even, f_{3n+3} is even.

Then $f_{3n+4} = f_{3n+3} + f_{3n+2}$. Since even + odd is odd, f_{3n+4} is odd.

Then $f_{3n+5} = f_{3n+4} + f_{3n+3}$. Since odd + even is odd, f_{3n+5} is odd.