2nd order

Finding general solution to LINEAR equation:

Solve $ay'' + by' + cy = g(t)$, $y(0) = y_0$, $y'(0) = y_1$

Step 1: Solve homogeneous eqn $ay''+by'+cy = 0$ (**)

Step 1a: Guess solution to (**): Suppose $y = e^{rt}$

$y = e^{rt}$ implies $y' = re^{rt}$ and $y'' = r^2e^{rt}$

$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$ implies $ar^2 + br + c = 0$,

Thus $r = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

Let $r_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $r_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$.

Thus $y = e^{r_1 t}$ and $y = e^{r_2 t}$ are both solutions to (**)

We will assume $r_1 \neq r_2$ so that we have two different solutions.

Step 1b: Find general soln to homogeneous eqn (**)

Note that (**) is a linear equation. Thus since $y = e^{r_1 t}$ and $y = e^{r_2 t}$ are both solutions to (**),

$y = c_1e^{r_1 t} + c_2e^{r_2 t}$ is the general solution to (**).

linear combination of sol'ns to linear homog eqn is also a sol'n

$=0$

Step 2: Solve $ay'' + by' + cy = g(t)$ (*)

Step 2a: Guess solution to (*).
Step 2b: Use thm below to form general soln to (*).

Thm: Suppose $c_1\phi_1(t) + c_2\phi_2(t)$ is a general solution to

$$ay'' + by' + cy = 0,$$

If $y(t) = \psi(t)$ is a solution to

$$ay'' + by' + cy = g(t) \; [*],$$

sol'n to non homg

Then $\psi(t) + c_1\phi_1(t) + c_2\phi_2(t)$ is also a solution to [*].

Step 3: If initial value problem:

Once general solution is known, can solve initial value problem (i.e., use initial conditions to find c_1, c_2):

General solution: $y(t) = \psi(t) + c_1\phi_1(t) + c_2\phi_2(t)$
Initial conditions: $y(0) = y_0$, $y'(0) = y_1$

Solve the following system of eqns for c_1 and c_2:

$$\left.\begin{aligned} y_0 &= \psi(0) + c_1\phi_1(0) + c_2\phi_2(0) \\ y_1 &= \psi'(0) + c_1\phi_1'(0) + c_2\phi_2'(0) \end{aligned}\right\}$$

solve using Cramer's Rule

6.4 Permutations with Forbidden Positions

Goal: To **derive** a more general formula for counting the number of permutations with arbitrary forbidden positions.

Recall in section 6.3, we looked at permutations with forbidden positions A derangement corresponds to non-attacking rook placement with forbidden positions along the diagonal (j, j), for $j = 1, ..., n$. In this section, we will cover arbitrary forbidden positions.

Let $X_j \subset \{1, ..., n\}$ for $j = 1, ..., n$.

Defn: $P(X_1, X_2, ..., X_n)$ = the set of permutations $i_1 i_2 ... i_n$ of $\{1, ..., n\}$ such that $i_j \notin X_j$.

Defn: $p(X_1, X_2, ..., X_n) = |P(X_1, X_2, ..., X_n)|$

Ex: $P(X_1, X_2, ..., X_n)$ corresponds to the set of derangements of $\{1, ..., n\}$ if $X_j = \{j\}$. Thus $D_n = |P(\{1\}, \{2\}, ..., \{n\}|$

Recall, we can visualize permutations with forbidden positions via $n \times n$ chessboards.

Ex: Derangements of $\{1, 2, 3\}$: $X_j = \{j\}$.

		X
	X	
X		

		X
	X	
X		

Non-derangement example:
$n = 4$, $X_i = \{j, j+1\}$, $j = 1, 2, 3$, $X_4 = \emptyset$.

		X	
	X	X	
X	X		
X			

		X	
	X	X	
X	X		
X			

		X	
	X	X	
X	X		
X			

		X	
	X	X	
X	X		
X			

$$P(X_1, X_2, ..., X_n) = P(\{1,2\}, \{2,3\}, \{3,4\}, \emptyset)$$
$$= \{3124, 3412, 3421, 4123\}.$$

$$p(X_1, X_2, ..., X_n) = p(\{1,2\}, \{2,3\}, \{3,4\}, \emptyset)$$
$$= |\{3124, 3412, 3421, 4123\}| = 4.$$

We can use the inclusion-exclusion principle to calculate $p(X_1, X_2, ..., X_n)$ (although in many cases, the computation can be tediously long and beyond computer capabilities for large n).

Thm 6.4.1:
$p(X_1, X_2, ..., X_n) = n! - r_1(n-1)! + r_2(n-2)! - ... + (-1)^n r_n$.

Proof (Similar to the proof of Thm 6.3.1.):

By the inclusion-exclusion principle,

$p(X_1, X_2, ..., X_n) =$
$|S| - \Sigma_{j=1}^{n} |A_j| + \Sigma_{i,j} |A_i \cap A_j| - ... + (-1)^n |A_1 \cap A_2 \cap ... \cap A_n|$

where

Let S = the set of permutations of $\{1, ..., n\}$. Then $|S| = n!$.

Let A_j = set of permutations $i_1 i_2 ... i_n$ such that $i_j \in X_j$ (for a fixed j).

r_i = # of ways to place i rooks in nonattacking positions and in forbidden positions special case of Thm 6.4.1

$\boxed{\text{Thm 6.3.1: For } n \geq 1,\ D_n = n!(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!})}$

Pf: Use the inclusion and exclusion principle: If $A_i \subset S$, $\overline{\cup A_i} = |S| - \Sigma_{j=1}^n |A_j| + \Sigma_{i,j} |A_i \cap A_j| - \ldots + (-1)^n |A_1 \cap A_2 \cap \ldots \cap A_n|$.

Choose S: What can we count which contains the set of derangements?

Let S = the set of permutations of $\{1, \ldots, n\}$. Then $|S| = n!$.

Choose A_j such that the set of derangements $= \overline{\cup A_j}$.
Let A_j = set of permutations such that j is in the jth spot.

$|A_j| = (n-1)!$ since there is only one choice for the jth spot (namely j), leaving $n-1$ terms to permute in the remaining $n-1$ places.

$|A_i \cap A_j| = (n-2)!$ since there is only one choice for the ith spot (namely i) and only one choice for the jth spot (namely j), leaving $n-2$ terms to permute in the remaining $n-2$ places.

Similarly, $|A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}| = (n-k)!$.

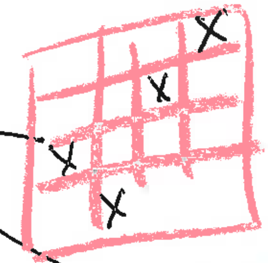
$$\boxed{\text{Thus } D_n = n! - \Sigma_{j=1}^n (n-1)! + \Sigma_{i,j} (n-2)! - \ldots + (-1)^n (n-n)!}$$

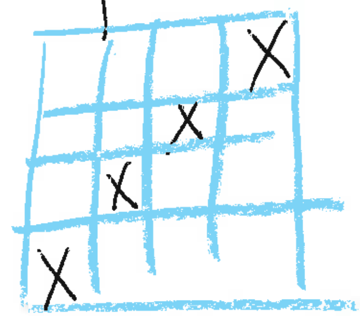
$$\boxed{= \binom{n}{0} n! - \binom{n}{1} (n-1)! + \binom{n}{2} (n-2)! - \ldots + \binom{n}{n} (-1)^n 0!}$$

$$\boxed{= n! - \frac{n!}{1!} + \frac{n!}{2!} + \ldots + (-1)^n \frac{n!}{n!} = n!(1 - \frac{1}{1!} + \frac{1}{2!} + \ldots + (-1)^n \frac{1}{n!})}$$

Recall $\binom{n}{k}$ = number of ways to choose k A_i's.

EX NOT DERANGEMENT





no conflicts among forbidden positions

non-attacking

$\Rightarrow$ only 1 forbidden position per row & column

$\Rightarrow r_i = \binom{n}{i}$

Note there are $|X_j|$ ways to place a rook in the jth position. There are $(n-1)!$ ways to place the remaining $n-1$ rooks so that the permutation belongs to A_j.

Thus $|A_j| = |X_j|(n-1)!$.

$$\Sigma_{j=1}^{n}|A_j| = \Sigma_{j=1}^{n}|X_j|(n-1)! = (n-1)!\Sigma_{j=1}^{n}|X_j| = r_1(n-1)!$$

where $r_1 = \Sigma_{j=1}^{n}|X_j|$.

Note r_1 = number of ways to place 1 nonattacking rooks on an $n \times n$ chessboard so that the rook is in a forbidden position.

Let's now look at $A_j \cap A_k$. $i_1 i_2 ... i_n \in A_j \cap A_k$, then $i_j \in X_j$ and $i_k \in X_k$. Thus there are $|X_j|$ ways to place a rook in the jth position and $|X_k|$ ways to place a rook in the kth position. There are $(n-2)!$ ways to place the remaining $n-1$ rooks so that the permutation belongs to $A_j \cap A_k$.

Thus $|A_j \cap A_k| = \cancel{|X_j||X_k|}(n-2)!$.

$\Sigma_{i,j}|A_i \cap A_j| = \Sigma_{i,j}\cancel{|X_j||X_k|}(n-2)! = (n-2)!\Sigma_{i,j}\cancel{|X_j||X_k|}$. ρ_{jk} ρ_{jk}

~~Let $r_2 = \Sigma_{i,j}|X_j||X_k|$.~~

Let ~~Note~~ r_2 = number of ways to place 2 nonattacking rooks on an $n \times n$ chessboard so that each of the 2 rooks is in a forbidden position.

Similarly, define r_k = number of ways to place k nonattacking rooks on an $n \times n$ chessboard so that each of the k rooks is in a forbidden position.

Then $\Sigma|A_{i_1} \cap A_{i_2} \cap ... \cap A_{i_k}| = r_k(\cancel{k-1})!$. □

$= r_k(n-k)!$

Note that if there are many forbidden positions, then r_k may be difficult to calculate and it may be easier to calculate $p(X_1, X_2, ..., X_n)$ directly. If there are few forbidden positions, Thm 6.4.1 is the easier method to compute $p(X_1, X_2, ..., X_n)$.

Example: Let $X = \{1, 2, 3\}$. Let $i_1 i_2 i_3 \in P(\{1,2\},\{1,3\},\{3\})$. Then there is only one choice for both i_1 and i_2, namely $i_1 = 3$ and $i_2 = 2$. But this leaves only one choice for $i_3 = 1$.

Thus $p(\{1,2\},\{1,3\},\{3\}) = 1$.

O	X	X
X	O	
X	X	O

Note in this case, it was easiest to count directly and not use Thm 6.4.1.

Example: Let $X = \{1, 2, 3, 4, 5, 6, 7\}$.

Calculate $p(\{1,2\},\{1,3\},\{3\},\{5\},\{4\},\emptyset,\emptyset)$

7 x 7 chessboard

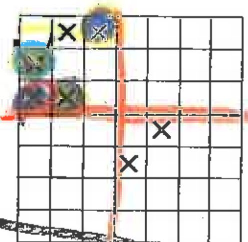
$n = 7$

	X	X				
X						
X	X					
				X		
			X			

r_1 = the number of forbidden positions = 7.

place 2 rooks in nonattacking position in forbidden positions

To calculate r_i for $i > 1$, note that the forbidden positions can be partitioned into two independent sets. Let $F_1 =$ the forbidden positions in the upper left 3×3 corner. Let F_2 contain the other two forbidden positions in a 2×2 square. These positions are independent because a rook in F_1 cannot attack a rook in F_2 (and vice versa).

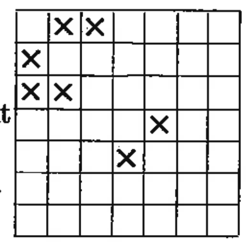


3+2+1

To calculate r_2 we break the problem into the following cases:

Case 1: both rooks are in F_1.

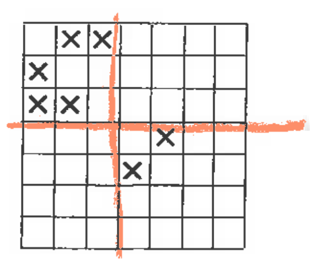
Subcase 1: one rook is placed in the 3rd column. There is only 1 possible placement for a rook in the 3rd column (1st row). There are 3 possible placements for the second rook so that the two rooks are in F_1 in non-attacking position.



Subcase 2: both rooks are place in the first two columns. In this case, the placement possibilities correspond to 13, 21, 23. Thus there are 3 possibilities in this subcase.

Thus there are $3 + 3 = 6$ possible non-attacking rook placements when both rooks are in F_1.

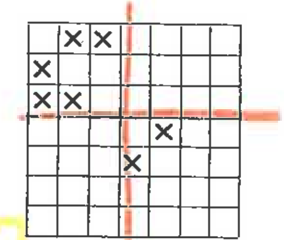
Case 2: both rooks are in F_2.



Since there are only 2 rooks in F_2, there is only one way to place both rooks in F_2.

1

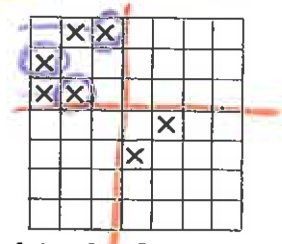
Case 3: one rook is in F_1 while the other rook is in F_2.



There are $5 \times 2 = \cancel{7}$ 10 ways to place one rook in F_1 and one rook in F_2.

Thus $r_2 = 6 + 1 + \cancel{7}\ 12 = \cancel{14}$ 17

To calculate r_3 we break the problem into the following cases:



Case 1: all 3 rooks are in F_1:

There is only 1 possible placement for a rook in the 3rd column (1st row). Thus in the second column, the rook must be placed in the first row. Thus in the first column, the remaining rook must be placed in the second row

Thus the only valid placement corresponds to the permutation 213

1

Case 2: 2 rooks in F_1, 1 rook in F_2.

By above, there are 6 ways to place 2 rooks in F_1 and 2 ways to place 1 rook in F_2. Thus there are $(6)(2) = 12$ ways to place 2 rooks in F_1, 1 rook in F_2.

Case 3: 1 rook in F_1, 2 rooks in F_2.

By above, there are 5 ways to place 1 rook in F_1 and 1 way to place 2 rook in F_2. Thus there are $(5)(1) = 5$ ways to place 1 rooks in F_1, 2 rook in F_2.

Hence $r_3 = 1 + 12 + 5 = 18$

To calculate r_4 we break the problem into the following cases:

Case 1: 3 rooks in F_1, 1 rook in F_2: ← calculated for r_3

By above, there is 1 way to place 3 rooks in F_1 and 2 ways to place 1 rook in F_2. Thus there are $(1)(2) = 2$ ways to place 3 rooks in F_1, 1 rook in F_2.

Case 2: 2 rooks in F_1, 2 rook in F_2. ← calculated for r_2

By above, there are 6 ways to place 2 rooks in F_1 and 1 way to place 1 rook in F_2. Thus there are $(6)(1) = 6$ ways to place 2 rooks in F_1, 2 rook in F_2.

Hence $r_4 = 2 + 6 = 8$

To calculate r_5 we note that if we have 5 nonattacking rooks in forbidden positions, 3 rooks are in F_1 and 2 rook in F_2. By above, there is 1 way to place 3 rooks in F_1 and 1 way to place 2 rook in F_2. Thus there are $(1)(1) = 1$ way to place 3 rooks in F_1, 2 rook in F_2. **Thus** $r_5 = 1$.

Note $r_6 = r_7 = 0$ as we can't place more than 5 nonattacking rooks in forbidden positions if we only have 5 columns which contain forbidden positions.

Hence $p(\{1,2\},\{1,3\},\{3\},\{5\},\{4\},\emptyset,\emptyset)$ $n = 7$

$$= n! - r_1(n-1)! + r_2(n-2)! - \ldots + (-1)^n r_n$$

$$= 7! - r_1(6!) + r_2(5!) - r_3(4!) + r_4(3!) - r_5(2!)$$

$$= 7! - 7(6!) + \cancel{14}(5!) - 18(4!) + 8(3!) - (2!).$$

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$h_{100} = 3 + 100(5)$

7.1: Sequences

Arithmetic sequence: $h_0, h_0 + q, h_0 + 2q, \ldots$

$h_n = h_{n-1} + q = h_0 + nq,\ n \geq 0$ ← h_0

Example: $h_n = 3 + 5n$: 3, 8, 13, 18, 23, 28, ...

Geometric sequence: $h_0, qh_0, q^2h_0, \ldots$

$h_n = qh_{n-1} = q^n h_0,\ n \geq 0$

Example: $h_n = 2^n$: $1, 2, 4, 8, 16, 32, 62, 128, 256, 512, \ldots$

$h_n = 2^n$ = number of combinations of an n-element set.

Partial sums: $s_n = \sum_{k=0}^{n} h_k$

Partial sums of arithmetic sequence:

$$s_n = \sum_{k=0}^{n} h_0 + kq = \sum_{k=0}^{n} h_0 + \sum_{k=0}^{n} kq = (n+1)h_0 + \frac{qn(n+1)}{2}$$

Example: If $h_k = 3+5k$, then $s_n = \sum_{k=0}^{n} h_k = (n+1)3 + \frac{5n(n+1)}{2}$

3, 11, 24, 42, 65, 93, ...

Geometric sequence: $s_n = \sum_{k=0}^{n} q^k h_0 = \begin{cases} \frac{q^{n+1}-1}{q-1} h_0 & q \neq 1 \\ (n+1)h_0 & q = 1 \end{cases}$

Example: If $h_k = 2^k$, then $s_n = \sum_{k=0}^{n} h_k = \frac{2^{n+1}-1}{2-1}$

1, 3, 7, 15, 31, 63, ...

Fibonacci:

$f_0 = 0,\ f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$

Suppose a pair of rabbits of the opposite sex give birth to a pair of rabbits of opposite sex every month starting with their second month. If we begin with a pair of newly born rabbits, how many rabbits are there after one year.

Let f_n = # of pairs of rabbits at the beginning of month n

$f_0 = 0 \quad f_1 = 1 \quad f_2 = 1 \quad f_3 = 2 \quad f_4 = 3 \quad f_5 = 3 + 2$

Hence $f_n = f_{n-1} + f_{n-2}$ ← 2 months have passed

Lemma: $s_n = \sum_{k=0}^{n} f_n = f_{n-2} - 1$

Proof by induction on n.

Lemma: f_n is even iff $3|n$.

Proof by induction on n.

Note that $f_0 = 0$ is even, $f_1 = 1$ is odd, and $f_2 = 1$ is odd. ← back ck

Suppose f_{3n} is even, f_{3n+1} is odd, and f_{3n+2} is odd.

Then $f_{3n+3} = f_{3n+2} + f_{3n+1}$. Since odd + odd is even, f_{3n+3} is even.

Then $f_{3n+4} = f_{3n+3} + f_{3n+2}$. Since even + odd is odd, f_{3n+4} is odd.

Then $f_{3n+5} = f_{3n+4} + f_{3n+3}$. Since odd + even is odd, f_{3n+5} is odd.