6.4 Permutations with Forbidden Positions

Goal: To derive a more general formula for counting the number of permutations with arbitrary forbidden positions.

Recall in section 6.3, we looked at permutations with forbidden positions A derangement corresponds to non-attacking rook placement with forbidden positions along the diagonal (j,j), for $j = 1,...,n$. In this section, we will cover arbitrary forbidden positions.

Let $X_j \subset \{1,...,n\}$ for $j = 1,...,n$.

Defn: $P(X_1, X_2,...,X_n)$ = the set of permutations $i_1 i_2 ... i_n$ of $\{1,...,n\}$ such that $i_j \notin X_j$.

(jth column)

Defn: $p(X_1, X_2,...,X_n) = |P(X_1, X_2,...,X_n)|$

Ex: $P(X_1, X_2,...,X_n)$ corresponds to the set of derangements of $\{1,...,n\}$ if $X_j = \{j\}$. Thus $D_n = |P(\{1\},\{2\},...,\{n\})|$

Recall, we can visualize permutations with forbidden positions via $n \times n$ chessboards.

Ex: Derangements of $\{1,2,3\}$:
$X_j = \{j\}$.

Non-derangement example:

$n = 4$, $X_j = \{j, j+1\}$, $j = 1,2,3$, $X_4 = \emptyset$.

3124   3412   3421   4123

$P(X_1, X_2,...,X_n) = P(\{1,2\},\{2,3\},\{3,4\},\emptyset)$
$= \{3124, 3412, 3421, 4123\}$.

$p(X_1, X_2,...,X_n) = p(\{1,2\},\{2,3\},\{3,4\},\emptyset)$
$= |\{3124, 3412, 3421, 4123\}| = 4$.

We can use the inclusion-exclusion principle to calculate $p(X_1, X_2,...,X_n)$ (although in many cases, the computation can be tediously long and beyond computer capabilities for large n).

Thm 6.4.1:
$$p(X_1, X_2,...,X_n) = n! - r_1(n-1)! + r_2(n-2)! - ... + (-1)^n r_n.$$

Proof (Similar to the proof of Thm 6.3.1):

By the inclusion-exclusion principle,

$p(X_1, X_2,...,X_n) =$
$|S| - \sum_{j=1}^n |A_j| + \sum_{i,j}|A_i \cap A_j| - ... + (-1)^n |A_1 \cap A_2 \cap ... \cap A_n|$

where

Let S = the set of permutations of $\{1,...,n\}$. Then $|S| = n!$.

Let A_j = set of permutations $i_1 i_2 ... i_n$ such that $i_j \in X_j$.
(for a fixed j).

where r_i = the # of ways to place i rooks in nonattacking position in forbidden position

$P(\{1\}, \{2\}, \{3\}) = $ set of Derangement = Derangement

Note that if there are many forbidden positions, then r_k may be difficult to calculate and it may be easier to calculate $p(X_1, X_2, ..., X_n)$ directly. If there are few forbidden positions, Thm 6.4.1 is the easier method to compute $p(X_1, X_2, ..., X_n)$.

Example: Let $X = \{1, 2, 3\}$. Let $i_1 i_2 i_3 \in P(\{1,2\}, \{1,3\}, \{3\})$. Then there is only one choice for both i_1 and i_2, namely $i_1 = 3$ and $i_2 = 2$. But this leaves only one choice for $i_3 = 1$.

Thus $p(\{1,2\}, \{1,3\}, \{3\}) = 1$.

Note in this case, it was easiest to count directly and not use Thm 6.4.1.

Example: Let $X = \{1, 2, 3, 4, 5, 6, 7\}$.

Calculate $p(\{1,2\}, \{1,3\}, \{3\}, \{5\}, \{4\}, \emptyset, \emptyset)$

$r_1 =$ the number of forbidden positions $= 7$.

Note there are $|X_j|$ ways to place a rook in the jth position. There are $(n-1)!$ ways to place the remaining $n-1$ rooks so that the permutation belongs to A_j.

Thus $|A_j| = |X_j|(n-1)!$.

$\sum_{j=1}^n |A_j| = \sum_{j=1}^n |X_j|(n-1)! = (n-1)!\sum_{j=1}^n |X_j| = r_1(n-1)!$

where $r_1 = \sum_{j=1}^n |X_j|$.

(# of forbidden positions)

Note $r_1 =$ number of ways to place 1 nonattacking rooks on an $n \times n$ chessboard so that the rook is in a forbidden position.

Let's now look at $A_j \cap A_k$. $i_1 i_2 ... i_n \in A_j \cap A_k$, then $i_j \in X_j$ and $i_k \in X_k$. Thus there are $|X_j|$ ways to place a rook in the jth position and $|X_k|$ ways to place a rook in the kth position. There are $(n-2)!$ ways to place the remaining $n-1$ rooks so that the permutation belongs to $A_j \cap A_k$.

Thus $|A_j \cap A_k| = |X_j||X_k|(n-2)!$.   *(not independent events)*

$\sum_{i,j} |A_i \cap A_j| = \sum_{i,j}|X_j||X_j|(n-2)! = (n-2)!\sum_{i,j}|X_j||X_k|$.   *(we choose columns i,j)* $P(3,2)$

Let $r_2 = \sum_{i,j}|X_j||X_k|$.

Note $r_2 =$ number of ways to place 2 nonattacking rooks on an $n \times n$ chessboard so that each of the 2 rooks is in a forbidden position.   *(painful to calculate usually)*

Similarly, define $r_k =$ number of ways to place k nonattacking rooks on an $n \times n$ chessboard so that each of the k rooks is in a forbidden position.

Then $\sum |A_{i_1} \cap A_{i_2} \cap ... \cap A_{i_k}| = r_k(n-k)!$.

$= r_k \underbrace{(n-k)!}$ *place remaining $n-k$ rooks in remaining $n-k$ rows*

place k rooks in forbidden position

To calculate r_i for $i > 1$, note that the forbidden positions can be partitioned into two independent sets. Let F_1 = the forbidden positions in the upper left 3×3 corner. Let F_2 contain the other two forbidden positions in a 2×2 square. These positions are independent because a rook in F_1 cannot attack a rook in F_2 (and vice versa).
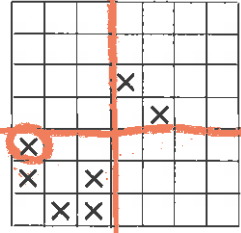
To calculate r_2 we break the problem into the following cases:

Case 1: both rooks are in F_1.

Subcase 1: one rook is placed in the 3rd column. There is only 1 possible placement for a rook in the 3rd column (1st row). There are 3 possible placements for the second rook so that the two rooks are in F_1 in non-attacking position.
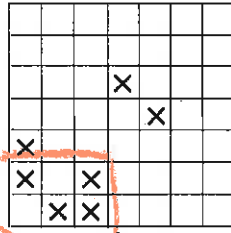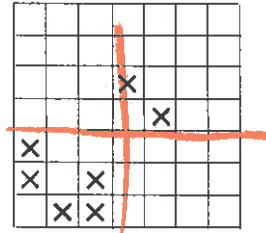

Subcase 2: both rooks are place in the first two columns. In this case, the placement possibilities correspond to 13, 21, 23. Thus there are 3 possibilities in this subcase.

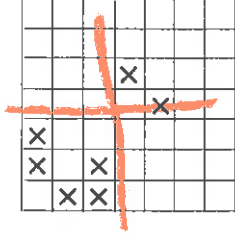Thus there are $3 + 3 = 6$ possible non-attacking rook placements when both rooks are in F_1.

Case 2: both rooks are in F_2.

Since there are only 2 rooks in F_2, there is only one way to place both rooks in F_2.

Case 3: one rook is in F_1 while the other rook is in F_2.

There are $5 + 2 = 7$ ways to place one rook in F_1 and one rook in F_2.

Thus $r_2 = 6 + 1 + 7 = 14$.

To calculate r_3 we break the problem into the following cases:

Case 1: all 3 rooks are in F_1:

There is only 1 possible placement for a rook in the 3rd column (1st row). Thus in the second column, the rook must be placed in the first row. Thus in the first column, the remaining rook must be placed in the second row
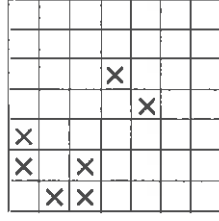
Thus the only valid placement corresponds to the permutation 213

Case 2: 2 rooks in F_1, 1 rook in F_2.

By above, there are 6 ways to place 2 rooks in F_1 and 2 ways to place 1 rook in F_2. Thus there are $(6)(2) = 12$ ways to place 2 rooks in F_1, 1 rook in F_2.

Case 3: 1 rook in F_1, 2 rooks in F_2.

By above, there are 5 ways to place 1 rook in F_1 and 1 way to place 2 rook in F_2. Thus there are $(5)(1) = 5$ ways to place 1 rook in F_1, 2 rook in F_2.

Hence $r_3 = 1 + 12 + 5 = 18$