$D_7 = 6(D_{n-2} + D_{n-1}) = 6(44 + 265)$

Recall i_j = # in jth spot

We can derive a recursive formula for D_n (we will look at many recursive formulas in chapter 7).

Lemma A: $\boxed{D_n = (n-1)(D_{n-2} + D_{n-1})}$ for $n \geq 3$.

Note the above formula is a recursive formula as we can determine D_n by calculating D_k for $k < n$.

Note $D_1 = 0$, $D_2 = 1$ (as 2 1 is the only derangement of $\{1, 2\}$).

Thus $D_3 = 2(0+1) = 2$, $D_4 = 3(1+2) = 9$, $D_5 = 4(2+9) = 44$, etc.

$(3-1)(0+1)$ $(4-1)(1+2)$

$D_6 = 5(9+44) = 5(53) = 265$

Combinatorial proof of lemma A:

Let $\mathcal{D}_n$ = the set of derangements of $\{1, ..., n\}$.

D_n = the number of derangements of $\{1, ..., n\} = |\mathcal{D}_n|$.

We need to show that D_n is a product of $n-1$ and $D_{n-2} + D_{n-1}$. If we can partition $\mathcal{D}_n$ into $n-1$ subsets where each subset has $D_{n-2} + D_{n-1}$ elements, we can use the multiplication principle to show $D_n = (n-1)(D_{n-2} + D_{n-1})$.

We also want to relate D_n to D_{n-1} = the number of dearrangements of $\{1, ..., n-1\}$ (and D_{n-2}).

Let's focus on one of the positions of a derangement. The last (nth) position of our derangement can be anything except n. Thus there are $n-1$ choices for the last (nth) position. Note the factor $n-1$ appears in our formula.

Let $\mathcal{R}_k$ = the set of derangements of $\{1, ..., n\}$ where k is in the nth position for $k = 1, ..., n-1$.

$\mathcal{R}_k = i_1 i_2 \cdots i_{n-1} k \in \mathcal{D}_n$

$i_j \neq j$

$k \neq n$

$i_n = k \neq n$

Then $\mathcal{D}_n = \bigcup_{j=0}^{n-1} \mathcal{R}_j$

$R_i \cap R_k = \emptyset$ for $i \neq k$

Let $r_k = |\mathcal{R}_k|$ the number of derangements such that k is in the nth position.

Note that $r_1 = r_2 = ... = r_{n-1}$ (while $r_n = 0$).

Then $D_n = r_1 + ... + r_{n-1} = r_{n-1} + ... + r_{n-1} = (n-1)r_{n-1}$.

Thus we have (hopefully) simplified our problem to showing that $D_{n-2} + D_{n-1} = r_{n-1}$ = the number of derangements such that $n-1$ is in the nth position.

We need to partition the permutations in $\mathcal{R}_{n-1}$ into two sets, one with D_{n-2} elements and the other with D_{n-1} elements.

We can easily take care of D_{n-2}. The numbers $n-1$ and n do not appear in any derangement of $\{1, ..., n-2\}$. In $\mathcal{R}_{n-1}$, $n-1$ appears in the last position. We can take a look at the derangements in $\mathcal{R}_{n-1}$, such that n appears in the $(n-1)$st position. If we remove the nth and $(n-1)$st entries, we obtain a derangement in $\mathcal{D}_{n-2}$.

Ex: for $n = 5$, $23154 \in \mathcal{R}_{n-1} \to 231 \in \mathcal{D}_{n-2}$.

$= R_4$ $= \mathcal{D}_3$

Thus D_{n-2} = the number of derangements of $\mathcal{R}_{n-1}$ such that n is in the $(n-1)$st position (and by definition of $\mathcal{R}_{n-1}$, $n-1$ is in the nth position).

We can now look at the remaining derangements in $\mathcal{R}_{n-1}$ where n is not in the $(n-1)$st position.

$R_{n-1} = D_{n-2} + ?$

Let $P_n = ?$

$\frac{i_1}{1}$ $\frac{i_2}{2}$ $\frac{i_{n-2}}{n-2}$ $\frac{k}{n-1}$ $\frac{n-1}{n}$ position & $k \neq n$

$k \neq n-1$ since derangement $n-1$ position can't be $n-1$

$i_1 \ldots i_{n-1}$ $\cancel{n}$

$\downarrow$ ← replace n w/ $(n-1)$

$a_1 \ldots a_{n-1}$

Let $\mathcal{P}_n$ the set of derangement where $n-1$ is in the nth position and k is in the $(n-1)$st position for some $k \neq n, n-1$ (I.e, $k \leq n-2$).

We would like to show that $D_{n-1} = |\mathcal{P}_n|$ = the number of derangements of $\{1, ..., n\}$ such that $n-1$ is in the nth position and k is in the $(n-1)$st position for some $k \leq n-2$

Let $\mathcal{D}_{n-1}$ = the set of derangements of $\{1, ..., n-1\}$.

We would like to create a bijection from $\mathcal{P}_n$ to $\mathcal{D}_{n-1}$

Note that the differences between $\mathcal{P}_n$ and $\mathcal{D}_{n-1}$. A derangement in $\mathcal{P}_n$ has n terms, while a derangement in $\mathcal{D}_{n-1}$ has $n-1$ terms. Thus we need to remove a term to go from $\mathcal{P}_n$ to $\mathcal{D}_{n-1}$.

If $i_1 i_2 ... i_n \in \mathcal{P}_n$, then $i_n = n-1$ and $i_{n-1} = k$ for some $k \leq n-2$. Also $i_j = n$ for some j.

In $\mathcal{D}_{n-1}$, $i_{n-1} = k$ for some $k \leq n-2$ (by definition of derangement of $\{1, ..., n-1\}$, so we have no problems with the $(n-1)$st term.

However, we have the following differences between $\mathcal{P}_n$ and $\mathcal{D}_{n-1}$:

$i_1 i_2 ... i_n$ has n terms and
n appears somewhere in $i_1 i_2 ... i_n$, and
$i_n = n-1$, so the placement of $n-1$ doesn't vary.
We can fix this by removing the nth term and replacing $i_j = n$ with $i_j = n-1$

Let $i_1 i_2 ... i_n \in \mathcal{P}_n$. Then $i_n = n-1$ and $i_{n-1} = k$ for some $k \leq n-2$.

Create $a_1 a_2 ... a_{n-1}$, a derangement of $\{1, ..., n-1\}$ by

$$\text{let } a_l = \begin{cases} i_l & \text{if } i_l \neq n,\ 1 \leq l \leq n-1 \\ n-1 & \text{if } i_l = n \end{cases}$$

Ex: For $n = 5$, $25314 \in \mathcal{P}_n \to 2431 \in \mathcal{D}_{n-1}$.

This gives us a bijection between $\mathcal{P}_n$ and $\mathcal{D}_{n-1}$. Thus $D_{n-1} = |\mathcal{P}_n|$.

Thus we have shown that $D_n = (n-1)r_{n-1} = (n-1)(D_{n-2} + |\mathcal{P}_n|) = (n-1)(D_{n-2} + D_{n-1})$ for $n \geq 3$.

Another (simpler) recurrance relation:

Lemma B: $D_n = nD_{n-1} + (-1)^n$ for $n \geq 2$

Proof by induction on n.

$n = 2$: $D_2 = 1$ (use definition or Thm 6.3.1)
$2D_1 + (-1)^2 = 2(0) + 1 = 1$.
Thus $D_n = nD_{n-1} + (-1)^n$ holds for $n = 2$.

Suppose $D_{k-1} = (k-1)D_{k-2} + (-1)^{k-1}$.

By lemma A, $D_k = (k-1)D_{k-2} + (k-1)D_{k-1}$

By the induction hypothesis, $D_{k-1} = (k-1)D_{k-2} + (-1)^{k-1}$.
Thus $(k-1)D_{k-2} = D_{k-1} - (-1)^{k-1} = D_{k-1} + (-1)^k$

Thus $D_k = D_{k-1} + (-1)^k + (k-1)D_{k-1} = kD_{k-1} + (-1)^k$

6.4 Permutations with Forbidden Positions

Goal: To **derive** a more general formula for counting the number of permutations with arbitrary forbidden positions.

Recall in section 6.3, we looked at permutations with forbidden positions A derangement corresponds to non-attacking rook placement with forbidden positions along the diagonal (j, j), for $j = 1, ..., n$. In this section, we will cover arbitrary forbidden positions.

Let $X_j \subset \{1, ..., n\}$ for $j = 1, ..., n$.

Defn: $P(X_1, X_2, ..., X_n)$ = the set of permutations $i_1 i_2 ... i_n$ of $\{1, ..., n\}$ such that $i_j \notin X_j$.

j^{th} column

Defn: $p(X_1, X_2, ..., X_n) = |P(X_1, X_2, ..., X_n)|$

Ex: $P(X_1, X_2, ..., X_n)$ corresponds to the set of derangements of $\{1, ..., n\}$ if $X_j = \{j\}$. Thus $D_n = |P(\{1\}, \{2\}, ..., \{n\}|$

Recall, we can visualize permutations with forbidden positions via $n \times n$ chessboards.

Ex: Derangements of $\{1, 2, 3\}$: $X_j = \{j\}$.

		X
	X	
X		

		X
	X	
X		

$P(\{1,2\}, \{2,3\}, \{3,4\}, \emptyset)$

column 1 avoids row 1 & row 2

Non-derangement example:
$n = 4$, $X_i = \{j, j+1\}$, $j = 1, 2, 3$, $X_4 = \emptyset$.

		X	
	X	X	
X	X		
X			

		X	
	X	X	
X	X		
X			

		X	
	X	X	
X	X		
X			

		X	
	X	X	
X	X		
X			

$$P(X_1, X_2, ..., X_n) = P(\{1,2\}, \{2,3\}, \{3,4\}, \emptyset)$$
$$= \{3124, 3412, 3421, 4123\}.$$

$$p(X_1, X_2, ..., X_n) = p(\{1,2\}, \{2,3\}, \{3,4\}, \emptyset)$$
$$= |\{3124, 3412, 3421, 4123\}| = 4.$$

We can use the inclusion-exclusion principle to calculate $p(X_1, X_2, ..., X_n)$ (although in many cases, the computation can be tediously long and beyond computer capabilities for large n).

Thm 6.4.1:
$p(X_1, X_2, ..., X_n) = n! - r_1(n-1)! + r_2(n-2)! - ... + (-1)^n r_n$.

Proof (Similar to the proof of Thm 6.3.1.):

By the inclusion-exclusion principle,

$p(X_1, X_2, ..., X_n) =$
$|S| - \Sigma_{j=1}^{n} |A_j| + \Sigma_{i,j} |A_i \cap A_j| - ... + (-1)^n |A_1 \cap A_2 \cap ... \cap A_n|$

where

Let S = the set of permutations of $\{1, ..., n\}$. Then $|S| = n!$.

Let A_j = set of permutations $i_1 i_2 ... i_n$ such that $i_j \in X_j$ (for a fixed j).