

$$D_7 = 6(D_{n-2} + D_{n-1}) = 6(44 + 265)$$

Recall  $i_j = \#$  in  $j^{\text{th}}$  spot

$$R_i \cap R_k = \emptyset \text{ for } i \neq k$$

We can derive a recursive formula for  $D_n$  (we will look at many recursive formulas in chapter 7).

Lemma A:  $D_n = (n-1)(D_{n-2} + D_{n-1})$  for  $n \geq 3$ .

Note the above formula is a recursive formula as we can determine  $D_n$  by calculating  $D_k$  for  $k < n$ .

Note  $D_1 = 0, D_2 = 1$  (as 2 1 is the only derangement of  $\{1, 2\}$ ).

Thus  $D_3 = 2(0+1) = 2, D_4 = 3(1+2) = 9, D_5 = 4(2+9) = 44,$   
 etc.  $(3-1)(0+1) = 2, (4-1)(1+2) = 9, D_6 = 5(9+44) = 5(53) = 265$

Combinatorial proof of lemma A:

Let  $D_n =$  the set of derangements of  $\{1, \dots, n\}$ .  
 $D_n =$  the number of derangements of  $\{1, \dots, n\} = |D_n|$ .

We need to show that  $D_n$  is a product of  $n-1$  and  $D_{n-2} + D_{n-1}$ .  
 If we can partition  $D_n$  into  $n-1$  subsets where each subset has  $D_{n-2} + D_{n-1}$  elements, we can use the multiplication principle to show  $D_n = (n-1)(D_{n-2} + D_{n-1})$ .

We also want to relate  $D_n$  to  $D_{n-1}$  = the number of derangements of  $\{1, \dots, n-1\}$  (and  $D_{n-2}$ ).

Let's focus on one of the positions of a derangement. The last ( $n$ th) position of our derangement can be anything except  $n$ . Thus there are  $n-1$  choices for the last ( $n$ th) position. Note the factor  $n-1$  appears in our formula.

Let  $R_k =$  the set of derangements of  $\{1, \dots, n\}$  where  $k$  is in the  $n$ th position for  $k = 1, \dots, n-1$ .

$R_k = i_1 i_2 \dots i_{n-1} k \in D_n$   
 $i_j \neq j$   
 $k \neq n$   
 $i_n = k \neq n$

Then  $D_n = \bigcup_{j=0}^{n-1} R_j$

Let  $r_k = |R_k|$  the number of derangements such that  $k$  is in the  $n$ th position.

Note that  $r_1 = r_2 = \dots = r_{n-1}$  (while  $r_n = 0$ ).

Then  $D_n = r_1 + \dots + r_{n-1} = r_{n-1} + \dots + r_{n-1} = (n-1)r_{n-1}$ .  
 Thus we have (hopefully) simplified our problem to showing that  $D_{n-2} + D_{n-1} = r_{n-1}$  = the number of derangements such that  $n-1$  is in the  $n$ th position.

We need to partition the permutations in  $R_{n-1}$  into two sets, one with  $D_{n-2}$  elements and the other with  $D_{n-1}$  elements.

We can easily take care of  $D_{n-2}$ . The numbers  $n-1$  and  $n$  do not appear in any derangement of  $\{1, \dots, n-2\}$ . In  $R_{n-1}, n-1$  appears in the last position. We can take a look at the derangements in  $R_{n-1}$ , such that  $n$  appears in the  $(n-1)$ st position. If we remove the  $n$ th and  $(n-1)$ st entries, we obtain a derangement in  $D_{n-2}$ .

Ex: for  $n = 5, \underline{23154} \in R_{n-1} \rightarrow 231 \in D_{n-2}$ .  
 $R_4 = \emptyset_3$

Thus  $D_{n-2}$  = the number of derangements of  $R_{n-1}$  such that  $n$  is in the  $(n-1)$ st position (and by definition of  $R_{n-1}, n-1$  is in the  $n$ th position).

We can now look at the remaining derangements in  $R_{n-1}$  where  $n$  is not in the  $(n-1)$ st position.

$R_{n-1} = D_{n-2} + ?$   
 Let  $R_n = ?$

$i_1 \dots i_{n-1}$   
 $a_1 \dots a_{n-1}$   
 $\downarrow$  ← replace  $n$  w/  $(n-1)$

$\frac{i_1}{1} \frac{i_2}{2} \dots \frac{i_k}{k} \dots \frac{i_{n-1}}{n-1}$  Since derangement +  $n-1$  position can't be  $n-1$

Let  $\mathcal{P}_n$  the set of derangement where  $n-1$  is in the  $n$ th position and  $k$  is in the  $(n-1)$ st position for some  $k \neq n, n-1$  (i.e.  $k \leq n-2$ ).

We would like to show that  $D_{n-1} = |\mathcal{P}_n|$  = the number of derangements of  $\{1, \dots, n\}$  such that  $n-1$  is in the  $n$ th position and  $k$  is in the  $(n-1)$ st position for some  $k \leq n-2$

Let  $\mathcal{D}_{n-1}$  = the set of derangements of  $\{1, \dots, n-1\}$ .

We would like to create a bijection from  $\mathcal{P}_n$  to  $\mathcal{D}_{n-1}$

Note that the differences between  $\mathcal{P}_n$  and  $\mathcal{D}_{n-1}$ . A derangement in  $\mathcal{P}_n$  has  $n$  terms, while a derangement in  $\mathcal{D}_{n-1}$  has  $n-1$  terms. Thus we need to remove a term to go from  $\mathcal{P}_n$  to  $\mathcal{D}_{n-1}$ .

If  $i_1 i_2 \dots i_n \in \mathcal{P}_n$ , then  $i_n = n-1$  and  $i_{n-1} = k$  for some  $k \leq n-2$ . Also  $i_j = n$  for some  $j$ .

In  $\mathcal{D}_{n-1}$ ,  $i_{n-1} = k$  for some  $k \leq n-2$  (by definition of derangement of  $\{1, \dots, n-1\}$ ), so we have no problems with the  $(n-1)$ st term.

However, we have the following differences between  $\mathcal{P}_n$  and  $\mathcal{D}_{n-1}$ :

$i_1 i_2 \dots i_n$  has  $n$  terms and  $n$  appears somewhere in  $i_1 i_2 \dots i_n$ , and  $i_n = n-1$ , so the placement of  $n-1$  doesn't vary. We can fix this by removing the  $n$ th term and replacing  $i_j = n$  with  $i_j = n-1$

Let  $i_1 i_2 \dots i_n \in \mathcal{P}_n$ . Then  $i_n = n-1$  and  $i_{n-1} = k$  for some  $k \leq n-2$ .

Create  $a_1 a_2 \dots a_{n-1}$ , a derangement of  $\{1, \dots, n-1\}$  by

$$\text{let } a_l = \begin{cases} i_l & \text{if } i_l \neq n, 1 \leq l \leq n-1 \\ n-1 & \text{if } i_l = n \end{cases}$$

Ex: For  $n=5$ ,  $25314 \in \mathcal{P}_n \rightarrow 2431 \in \mathcal{D}_{n-1}$ .

This gives us a bijection between  $\mathcal{P}_n$  and  $\mathcal{D}_{n-1}$ . Thus  $D_{n-1} = |\mathcal{P}_n|$ .

Thus we have shown that  $D_n = (n-1)r_{n-1} = (n-1)(D_{n-2} + |\mathcal{P}_n|) = (n-1)(D_{n-2} + D_{n-1})$  for  $n \geq 3$ .

Another (simpler) recurrence relation:

Lemma B:  $D_n = nD_{n-1} + (-1)^n$  for  $n \geq 2$

Proof by induction on  $n$ .

$n=2$ :  $D_2 = 1$  (use definition or Thm 6.3.1)  $2D_1 + (-1)^2 = 2(0) + 1 = 1$ .

Thus  $D_n = nD_{n-1} + (-1)^n$  holds for  $n=2$ .

Suppose  $D_{k-1} = (k-1)D_{k-2} + (-1)^{k-1}$ .

By lemma A,  $D_k = (k-1)D_{k-2} + (k-1)D_{k-1}$

By the induction hypothesis,  $D_{k-1} = (k-1)D_{k-2} + (-1)^{k-1}$ . Thus  $(k-1)D_{k-2} = D_{k-1} - (-1)^{k-1} = D_{k-1} + (-1)^k$

Thus  $D_k = D_{k-1} + (-1)^k + (k-1)D_{k-1} = kD_{k-1} + (-1)^k$

$P(\{1,2\}, \{2,3\}, \{3,4\}, \emptyset)$   
 ↑ column 1 avoids row 1 & row 2

6.4 Permutations with Forbidden Positions

**Goal:** To derive a more general formula for counting the number of permutations with arbitrary forbidden positions.

Recall in section 6.3, we looked at permutations with forbidden positions. A derangement corresponds to non-attacking rook placement with forbidden positions along the diagonal  $(j, j)$ , for  $j = 1, \dots, n$ . In this section, we will cover arbitrary forbidden positions.

Let  $X_j \subset \{1, \dots, n\}$  for  $j = 1, \dots, n$ .

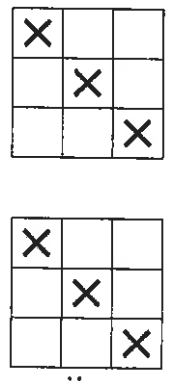
**Defn:**  $P(X_1, X_2, \dots, X_n)$  = the set of permutations  $i_1 i_2 \dots i_n$  of  $\{1, \dots, n\}$  such that  $i_j \notin X_j$ .

↑  $j$ th column

**Defn:**  $p(X_1, X_2, \dots, X_n) = |P(X_1, X_2, \dots, X_n)|$

**Ex:**  $P(X_1, X_2, \dots, X_n)$  corresponds to the set of derangements of  $\{1, \dots, n\}$  if  $X_j = \{j\}$ . Thus  $D_n = |P(\{1\}, \{2\}, \dots, \{n\})|$

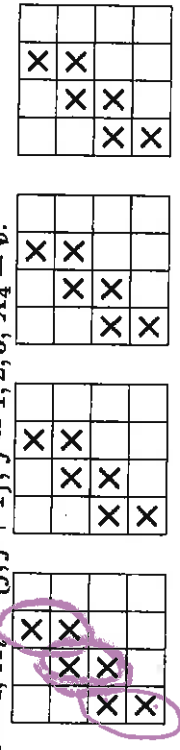
Recall, we can visualize permutations with forbidden positions via  $n \times n$  chessboards.



**Ex:** Derangements of  $\{1, 2, 3\}$ :  $X_j = \{j\}$ .

Non-derangement example:

$n = 4, X_i = \{j, j + 1\}, j = 1, 2, 3, X_4 = \emptyset$ .



$P(X_1, X_2, \dots, X_n) = P(\{1, 2\}, \{2, 3\}, \{3, 4\}, \emptyset)$

$= \{3124, 3412, 3421, 4123\}$ .

$p(X_1, X_2, \dots, X_n) = p(\{1, 2\}, \{2, 3\}, \{3, 4\}, \emptyset)$

$= |\{3124, 3412, 3421, 4123\}| = 4$ .

We can use the inclusion-exclusion principle to calculate  $p(X_1, X_2, \dots, X_n)$  (although in many cases, the computation can be tediously long and beyond computer capabilities for large  $n$ ).

**Thm 6.4.1:**

$p(X_1, X_2, \dots, X_n) = n! - r_1(n-1)! + r_2(n-2)! - \dots + (-1)^n r_n$ .

**Proof** (Similar to the proof of Thm 6.3.1.):

By the inclusion-exclusion principle,

$p(X_1, X_2, \dots, X_n) = |S| - \sum_{j=1}^n |A_j| + \sum_{i,j} |A_i \cap A_j| - \dots + (-1)^n |A_1 \cap A_2 \cap \dots \cap A_n|$

where

Let  $S$  = the set of permutations of  $\{1, \dots, n\}$ . Then  $|S| = n!$ .

Let  $A_j$  = set of permutations  $i_1 i_2 \dots i_n$  such that  $i_j \in X_j$  (for a fixed  $j$ ).