6.3 Derangements

Suppose each person in a group of n friends brings a gift to a party. In how many ways can the n gifts be distributed so that each person receives one gift and no person receives their own gift.

Let the set of friends $= \{p_1, ..., p_n\}$ where $p_j =$ person j.
Let the set of gifts $= \{g_1, ..., g_n\}$ where $g_j =$ the gift brought by person j.

Suppose $f : \{p_1, ..., p_n\} \to \{g_1, ..., g_n\}$,
$f(p_k) = g_j$ iff person p_k receives give g_j, the gift brought by person j.
If each person receives one gift, then f is a bijection.
If no person receives their own gift. Then $f(p_j) \neq g_j$.

In simpler notation,
$f : \{1, ..., n\} \to \{1, ..., n\}$ such that $f(j) \neq j$

Recall:
a permutation on $\{1, ..., n\}$ is a bijection $f : \{1, ..., n\} \to \{1, ..., n\}$

1 2 ... n
f(1) f(2) f(n)

Ex: The permutation 1 2 3 4 5 corresponds to the identity function.

$1 \to 1$, $2 \to 2$, ..., $5 \to 5$

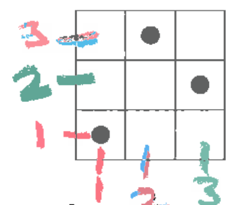
Ex: The permutation 1 3 2 corresponds to the function
$f(1) = 1$, $f(2) = 3$, $f(3) = 2$

Defn: A *derangement* of $\{1, ..., n\}$ is a permutation $i_1 i_2 ... i_n$ such that $i_j \neq j$. I.e, j is not in the jth place.

In function notation:
$f(j) = i_j$, then if $i_1 i_2 ... i_n$ is a derangement, $f(j) \neq j$.

In yet other wording, recall a permutation corresponds to the placement of n non-attacking rooks on an $n \times n$ chessboard.

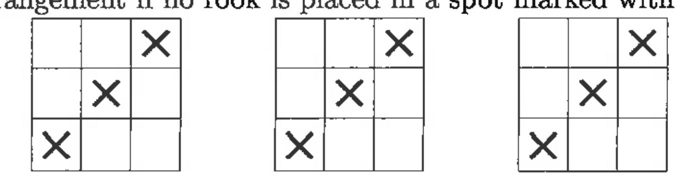
Ex: The permutation 1 3 2 corresponds to the following rook placement:



1 3 2
not a derangement

A derangement corresponds to non-attacking rook placement with forbidden positions along the diagonal (j, j), for $j = 1, ..., n$.

Ex: If rooks are placed on the following 3×3 chessboard in non-attacking position, then the rook placement corresponds to a derangement if no rook is placed in a spot marked with an X.



Thus the derangements of $\{1, 2, 3\}$ are 2 3 1 and 3 1 2.

Let $D_n =$ the number of derangements of $\{1, ..., n\}$.

Thus $D_3 = 2$.