$x_1 = 4 \quad x_2 = 2 \quad x_3 = 1 \quad x_4 = 0 \quad x_5 = 2$

$\underbrace{||||}_{x_1} + \underbrace{||}_{x_2} + \underbrace{|}_{x_3} + \underbrace{}_{x_4} + \underbrace{||}_{x_5}$

6.2: Combinations with repetitions.

What is the number of r−combinations of $\{n_1 \cdot 1,\ n_2 \cdot 2, ...,\ n_k \cdot k\}$?

If $k = 1$; The number of r−combinations of $\{n \cdot a\}$ = 1

If $r = \sum_{i=1}^{k} n_i$:
The # of r−combinations of $\{n_1 \cdot 1,\ n_2 \cdot 2, ...,\ n_k \cdot k\}$ = 1

If $r > \sum_{i=1}^{k} n_i$:
The # of r−combinations of $\{n_1 \cdot 1,\ n_2 \cdot 2, ...,\ n_k \cdot k\}$ = 0

If $n_i = 1\ \forall i$: ← not a multiset
The number of r−combinations of $\{1, 2, ..., k\}$ = $\binom{k}{r}$

If $n_i \geq r\ \forall i$: ← unlimited repeats
The number of r−combinations of $\{n_1 \cdot 1,\ n_2 \cdot 2, ...,\ n_k \cdot k\}$

= the number of integral solutions to $\sum_{i=1}^{k} x_i = r$

$x_1 + x_2 + \ldots + x_k$

= the number of permutations of $\{r \cdot 1, (k-1) \cdot +\}$

$$= \binom{r+k-1}{r} = \frac{(r+k-1)!}{r!(k-1)!}$$

What if $2 \leq n_i < r - 1$?

choosing: x_1 1's, x_2 2's, ..., x_k k's } r-subset

$x_1 + x_2 + \ldots + x_k = r$ elements all together

$k = 2$

Ex: The number of 10-combinations of $\{3 \cdot a,\ 9 \cdot b\}$ = 7

This problem is small enough to ~~break into cases~~ & use **sum rule:**

A 10-subset of $\{3 \cdot a,\ 9 \cdot b\}$ is of the form $\{n_1 \cdot a,\ n_2 \cdot b\}$ where

$0 \leq n_1 \leq 3;\ 0 \leq n_2 \leq 9;\ n_1 + n_2 = 10$

$n_1 = 0$: 0 $\quad n_1 = 1$: 1 $\quad n_1 = 2$: 1 $\quad n_1 = 3$: 1

Thus the number of 10-combinations of $\{3 \cdot a,\ 9 \cdot b\}$ = 3

$1 \leq n_1 \leq 3 \quad 7 \leq n_2 \leq 9$ — works for $k = 2$

Can also use inclusion-exclusion:

Let S = set of all 10-combinations of $\{\infty \cdot a,\ \infty \cdot b\}$, $|S| = 11$

Big set = remove all upper bounds

Let A_1 = set of all 10-combinations of $\{\infty \cdot a,\ \infty \cdot b\}$ containing more than 3 a's, $|A_1| = 7$

I.e., A_1 can have 4, 5, 6, 7, 8, 9, or 10 a's. $\quad 10 - 3 = 7.$

I.e., $|A_1|$ = the number of 6 combinations of $\{\infty \cdot a,\ \infty \cdot b\}$ = 7 $= \frac{(6+1)!}{6!1!}$

Let A_2 = set of all 10-combinations of $\{\infty \cdot a,\ \infty \cdot b\}$ containing more than 9 b's, $|A_2| = 1$

$A_1 \cap A_2$ = set of all 10-combinations of $\{\infty \cdot a,\ \infty \cdot b\}$ containing more than 3 a's ~~and~~ ~~OR~~ and more than 9 b's = 0

$\overline{A_1 \cup A_2}$ = The number of 10-combinations of $\{3 \cdot a,\ 9 \cdot b\}$

$$= \frac{(10+1)!}{10!1!} - (10-3) - (10-9) + 0 = 11 - 8 = 3$$

$k \geq 3$

What if $3 \le n_i \le r-1$? Use inclusion-exclusion (Usually).

Ex: The number of integral solutions to $\sum_{i=1}^{5} x_i = 20$
where $-2 \le x_i \le 7\ \forall i$

$=$ The number of integral solutions to $\sum_{i=1}^{5} y_i = 30$
where $0 \le y_i \le 9\ \forall i$

$=$ The number of 30-combinations of the multiset $\{9 \cdot a_1,\ 9 \cdot a_2,\ 9 \cdot a_3,\ 9 \cdot a_4,\ 9 \cdot a_5\}$

Pf: Let $y_i = x_i + 2$

See example in book for case when the number of a_i's is not the same $\forall i$.

Let $S =$ the set of integral solutions to $y_1 + y_2 + y_3 + y_4 + y_5 = 30$
where $0 \le y_i\ \forall i$

Then $|S| =$ the number of permutations of $\{30 \cdot 1, 4 \cdot +\} =$

For $i = 1, 2, 3, 4, 5$, (j)
let $A_i =$ the set of integral solutions to $y_1 + y_2 + y_3 + y_4 + y_5 = 30$
where $10 \le y_i$ (j)

$0 \le y_i\ \forall i \ne j$

Ex: $(10, 5, 5, 5, 5) \in A_1$, $(0, 20, 7, 2, 1) \in A_2$,
$(0, 0, 10, 10, 10) \in A_3 \cap A_4 \cap A_5$

Then $\overline{\cup_{i=1}^{5} A_i} =$ the set of of integral solutions to $\sum_{i=1}^{5} y_i = 30$
where $0 \le y_i \le 9\ \forall i$

Note $|A_1| = |A_2| = |A_3| = |A_4| = |A_5|$.

Note $|A_1| =$ The number of 30-combinations of the multiset $\{\infty \cdot a_1, \infty \cdot a_2, \infty \cdot a_3, \infty \cdot a_4, \infty \cdot a_5\}$ containing more than 9 a_1's.

The number of 30-combinations of the multiset $\{\infty \cdot a_1, \infty \cdot a_2, \infty \cdot a_3, \infty \cdot a_4, \infty \cdot a_5\}$ containing at least 10 a_1's.

$=$ The number of 20-combinations of the multiset $\{\infty \cdot a_1, \infty \cdot a_2, \infty \cdot a_3, \infty \cdot a_4, \infty \cdot a_5\}$

$=$ the set of integral solutions to $y_1 + y_2 + y_3 + y_4 + y_5 = 30$
where $y_1 \ge 10$ and $y_i \ge 0$ for $i = 2, 3, 4, 5$

$=$ the set of integral solutions to $z_1 + z_2 + z_3 + z_4 + z_5 = 20$
where $z_i \ge 0$ for $i = 1, 2, 3, 4, 5$

$=$

Note $|A_i \cap A_j| = |A_1 \cap A_2|$ for $i \ne j$.

The number of 30-combinations of the multiset $\{\infty \cdot a_1, \infty \cdot a_2, \infty \cdot a_3, \infty \cdot a_4, \infty \cdot a_5\}$
containing at least 10 a_1's and at least 10 a_2's

The number of 10-combinations of the multiset $\{\infty \cdot a_1, \infty \cdot a_2, \infty \cdot a_3, \infty \cdot a_4, \infty \cdot a_5\} =$

Etc. (see class notes).

Read book for examples where the pairwise intersections are not all identical