Defn: An *equivalence relation* is reflexive, symmetric, transitive.

Ex: $\cong_p$ is an equivalence relation where $x \cong_p y$ if $\frac{x-y}{p} \in \mathbb{Z}$

Claim: $\cong_p$ is reflexive. That is, $\forall x \in X$, $x \cong_p x$.

$\frac{x-x}{p} = 0 \in \mathbb{Z}$. Thus $x \cong_p x$.

Claim: $\cong_p$ is symmetric. I.e., if $x \cong_p y$, then $y \cong_p x$.

Suppose $x \cong_p y \Rightarrow \frac{x-y}{p} \in \mathbb{Z} \Rightarrow \frac{y-x}{p} \in \mathbb{Z} \Rightarrow y \cong_p x$

Claim: $\cong_p$ is transitive. I.e., if $x \cong_p y$ and $y \cong_p z$, then $x \cong_p z$.

Suppose $x \cong_p y$ & $y \cong_p z$

$\Rightarrow \frac{x-y}{p} = h \in \mathbb{Z}$ & $\frac{y-z}{p} = n \in \mathbb{Z}$

$\frac{x-y}{p} + \frac{y-z}{p} = \frac{x-z}{p} = h+n \in \mathbb{Z}$

$\Rightarrow x \cong_p z$

Thus $\cong_p$ is an equivalence relation.

Equivalence class $[a] = \{x \mid x \sim a\}$

For $\cong_2$

$[4] = \{\ldots, -4, -2, 0, 4, 2, 6, 8, 10, \ldots\} = [0]$

$[-2] =$ = set of even integers

$[1] =$ set of odd integers $= \{\ldots, -3, -1, 1, 3, \ldots\}$

Ex: $\mathbb{Z} = [0] \sqcup [1]$

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$\mathcal{P} = \{P_\alpha \mid \alpha \in A\}$ is a partition of X iff
$X = \bigcup_{P_\alpha \in \mathcal{P}} P_\alpha$, $P_\alpha \neq \emptyset \ \forall \alpha$, and $P_\alpha \cap P_\beta \neq \emptyset$ implies $P_\alpha = P_\beta$

Thm 4.5.3: If $\sim$ is an equivalence relation on X, then $\{[x_\alpha] \mid x_\alpha \in X\}$ is a partition of X.

If $\mathcal{P} = \{P_\alpha \mid \alpha \in A\}$ is a partition of X, then $x \sim y$ iff $\exists P_\alpha$ such that $x, y \in P_\alpha$ is an equivalence relation.

Proof: Suppose $\sim$ is an equivalence relation on X.

Claim: $\{[x_\alpha] \mid x_\alpha \in X\}$ is a partition of X.

Let $x_\alpha \in X$. Then $x_\alpha \in [x_\alpha]$ since $\sim$ is reflexive.
Thus $[x_\alpha] \neq \emptyset$ and $X = \bigcup_{x_\alpha \in X} [x_\alpha]$.

Suppose $[x_\alpha] \cap [x_\beta] \neq \emptyset$.

Claim: $[x_\alpha] = [x_\beta]$

Claim: $[x_\alpha] \subset [x_\beta]$ and $[x_\beta] \subset [x_\alpha]$

Claim: If $z \in [x_\alpha] = \{x \mid x \sim x_\alpha\}$, then $z \in [x_\beta] = \{x \mid x \sim x_\beta\}$

Proof of Claim: Since $z \in [x_\alpha]$, $z \sim x_\alpha$. Since

Thus $[x_\alpha] \subset [x_\beta]$. Similarly $[x_\beta] \subset [x_\alpha]$.

Suppose $\mathcal{P} = \{P_\alpha \mid a \in A\}$.

Claim: $x \sim y$ iff $\exists P_\alpha \in \mathcal{P}$ such that $x, y \in P_\alpha$ is an equivalence relation on X.

Proof of Claim: HW #44 (don't assume finite).

$\{(1243), (3124), (4312), (2431)\}$

$\parallel$

$[(1243)]$

12 34	31 42	43 21	24 13
12 43	41 32	34 21	23 14
13 24	21 43	42 31	34 12
13 42	41 23	24 31	32 14
14 23	21 34	32 41	21 34
14 32 $=_c$	31 24 $=_c$	23 41 $=_c$	42 13

Linear permutation of $\{1,2,3,4\}$ $\Rightarrow$ circular perm $\{1,2,3,4\}$

$$\frac{4!}{4} = 6$$ circular permutations

A circular permutation = an equivalence of linear permutation where equiv class defined by rotation or partition

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RB gg	gR gB	gg BR	Bg Rg
RB gg	gR gB	gg BR	Bg Rg
Rg Bg	BR gg	gB gR	gg RB
Rg gB	gR Bg	Bg gR	gB Rg
Rg Bg	BR gg	gB gR	BR gg
Rg gB	gR Bg	Bg gR	gB Rg

partial order:

$x \not\leq y \Rightarrow y \leq x$

$\leq$ reflexive

anti-symmetric $x \leq y$ & $y \leq x \Rightarrow x = y$

transitive

$<$ anti-reflexive

Thm 4.5.1: Suppose $|X| = n$. ~~Then there exists a bijection between~~ the total orders of X and the permutations of X. Hence there exists $n!$ different total orders on n.

Proof: Suppose $X = \{1, ..., n\}$ and suppose $f(1), f(2), ..., f(n)$ is a permutation of the elements of X. f is a bijection

Claim: $f(1) \leq f(2) \leq \ ... \ \leq f(n)$ defines a total order.

Note the above claim is equivalent to:

Claim: $f(i) \leq f(j)$ iff $i \leq j$ defines a total order on X.

Proof of claim:

Claim: $\leq$ is reflexive. That is, $\forall x \in X$, $x \leq x$.

$f(i) \leq f(i)$ since $i \leq i$

Claim: $\leq$ is anti-symmetric. I.e., if $x \leq y$ and $y \leq x$, then $x = y$.

Suppose $f(i) \leq f(j)$ & $f(j) \leq f(i)$

$\Rightarrow i \leq j$ & $j \leq i \Rightarrow i = j$

$\Rightarrow f(i) = f(j)$

Claim: $\leq$ is transitive. That is, if $x \leq y$ and $y \leq z$, then $x \leq z$.

Suppose $f(i) \leq f(j)$ & $f(j) \leq f(k)$

$\Rightarrow i \leq j$ & $j \leq k$

$\Rightarrow i \leq k \Rightarrow f(i) \leq f(k)$

Thus $\leq$ is a partial order. Note every pair of elements of X is comparable. Thus $\leq$ is a total order.

because f is a bijection so all element of X are listed in $f(1) \leq f(2) \leq \ ... \ \leq f(n)$

Suppose we have a total order $\leq$ on X.

Claim: We can order the elements of X as follows:

$f(1) \leq f(2) \leq ... \leq f(n)$ for some permutation of X.

Proof by induction on $n = |X|$.

Suppose $n = 1$:

Suppose that if $|X| = n - 1$, we can order the elements of X as follows: $f(1) < f(2) < ... < f(n-1)$ for some permutation of X.

Suppose $|X| = n$.

More elegant to say

$x = f(i) \leq f(j) = y$

Note that we have shown a 1:1 correspondence between permutations of X and total orders of X. Hence there exists $n!$ different total orders on n.

Alt pf: Suppose $x, y \in X \Rightarrow$

$\exists\, i, j \in \{1, .., n\}$ st $f(i) = x$ & $f(j) = y$

If $i \leq j$ $f(i) = x \leq y \leq f(j)$. If $j \leq i \Rightarrow f(j) = y \leq x \leq f(i)$