Defn: An *equivalence relation* is reflexive, symmetric, transitive.

Ex: $\cong_p$ is an equivalence relation where $x \cong_p y$ if $\frac{x-y}{p} \in \mathbb{Z}$

Claim: $\cong_p$ is reflexive. That is, $\forall x \in X$, $x \cong_p x$.

$\frac{x-x}{p} = 0 \in \mathbb{Z}$. Thus $x \cong_p x$.

Claim: $\cong_p$ is symmetric. I.e., if $x \cong_p y$, then $y \cong_p x$.

Suppose $x \cong_p y \Rightarrow \frac{x-y}{p} \in \mathbb{Z} \Rightarrow \frac{y-x}{p} \in \mathbb{Z} \Rightarrow y \cong_p x$

Claim: $\cong_p$ is transitive. I.e., if $x \cong_p y$ and $y \cong_p z$, then $x \cong_p z$.

Suppose $x \cong_p y$ & $y \cong_p z$

$\Rightarrow \frac{x-y}{p} = h \in \mathbb{Z}$ & $\frac{y-z}{p} = n \in \mathbb{Z}$

$\frac{x-y}{p} + \frac{y-z}{p} = \frac{x-z}{p} = h+n \in \mathbb{Z}$

$\Rightarrow x \cong_p z$

Thus $\cong_p$ is an equivalence relation.

Equivalence class $[a] = \{x \mid x \sim a\}$

For $\cong_2$

$[4] =$

$[-2] =$

$[1] =$

Ex: $\mathbb{Z} =$

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$\mathcal{P} = \{P_\alpha \mid \alpha \in A\}$ is a partition of X iff
$X = \bigcup_{P_\alpha \in \mathcal{P}} P_\alpha$, $P_\alpha \neq \emptyset\ \forall \alpha$, and $P_\alpha \cap P_\beta \neq \emptyset$ implies $P_\alpha = P_\beta$

Thm 4.5.3: If $\sim$ is an equivalence relation on X, then $\{[x_\alpha] \mid x_\alpha \in X\}$ is a partition of X.

If $\mathcal{P} = \{P_\alpha \mid \alpha \in A\}$ is a partition of X, then $x \sim y$ iff $\exists P_\alpha$ such that $x, y \in P_\alpha$ is an equivalence relation.

Proof: Suppose $\sim$ is an equivalence relation on X.

Claim: $\{[x_\alpha] \mid x_\alpha \in X\}$ is a partition of X.

Let $x_\alpha \in X$. Then $x_\alpha \in [x_\alpha]$ since $\sim$ is reflexive.
Thus $[x_\alpha] \neq \emptyset$ and $X = \bigcup_{x_\alpha \in X} [x_\alpha]$.

Suppose $[x_\alpha] \cap [x_\beta] \neq \emptyset$.

Claim: $[x_\alpha] = [x_\beta]$

Claim: $[x_\alpha] \subset [x_\beta]$ and $[x_\beta] \subset [x_\alpha]$

Claim: If $z \in [x_\alpha] = \{x \mid x \sim x_\alpha\}$, then $z \in [x_\beta] = \{x \mid x \sim x_\beta\}$

Proof of Claim: Since $z \in [x_\alpha]$, $z \sim x_\alpha$. Since

Thus $[x_\alpha] \subset [x_\beta]$. Similarly $[x_\beta] \subset [x_\alpha]$.

Suppose $\mathcal{P} = \{P_\alpha \mid a \in A\}$.

Claim: $x \sim y$ iff $\exists P_\alpha \in \mathcal{P}$ such that $x, y \in P_\alpha$ is an equivalence relation on X.

Proof of Claim: HW #44 (don't assume finite).

$\{(1243), (3124), (4312), (2431)\}$

$|||$

$[(1243)]$

12 34	31 42	43 21	24 13
12 43	41 32	34 21	23 14
13 24	21 43	42 31	34 12
13 42	41 23	24 31	32 14
14 23	21 34	32 41	21 34
14 32 $=_c$	31 24 $=_c$	23 41 $=_c$	42 13

Linear permutation of $\{1,2,3,4\}$ $\Rightarrow$ circular perm $\{1,2,3,4\}$

$\frac{4!}{4} = 6$ circular permutations

RB gg	gR gB	gg BR	Bg Rg
RB gg	gR gB	gg BR	Bg Rg
Rg Bg	BR gg	gB gR	gg RB
Rg gB	gR Bg	Bg gR	gB Rg
Rg Bg	BR gg	gB gR	BR gg
Rg gB	gR Bg	Bg gR	gB Rg