http://mail.baylorschool.org/~dkennedy/Numb3rs.ppt

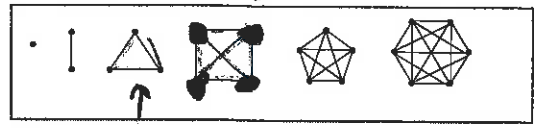
***NUMB3RS* Activity: A Party of Six**

Episode: "Protest"

Topic: Graph Theory and Ramsey Numbers
Grade Level: 8 - 12
Objective: To see how a complete graph with edges of two colors can be used to model acquaintances and non-acquaintances at a party.
Time: About 30 minutes
Materials: Red and blue pencils or markers, paper

The first six complete graphs:

K_2 K_4



K_1 K_3

(triangle

People = vertices

If two people (A and B) are at a party, there are only two possibilities: either A and B know each other, or A and B do not know each other. Draw the two possible graphs below.

color edge blue if A knows B & B knows A

color edge red if A & B don't know each other

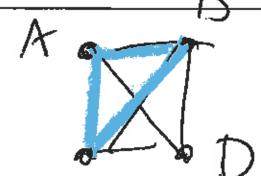
Draw all of the possible 3-person party graphs for A, B, and C below.

All possible colorings of K_3 using red & blue

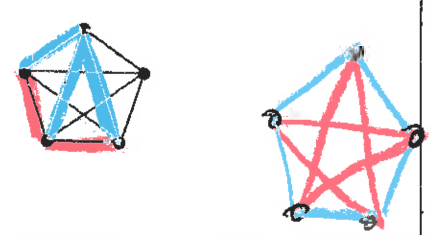
← A doesn't know B OR B " " A

There are 64 possible 4-person party graphs for guests A, B, C, and D (Why?), but you will not be asked to draw them all. Instead, draw the 8 possible 4-person party graphs in which A, B, and C all know each other. We say A, B, and C are *mutual acquaintances*.

not a vertex



It is actually possible to color the edges of a 5-person party graph in such a way that there are neither three people that are mutual acquaintances nor three people that are mutual non-acquaintances. Can you do it?



$r(3,3) > 5$

red K_3 blue K_3

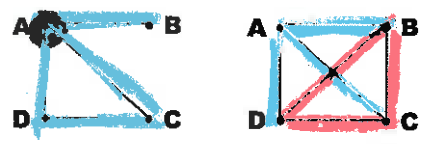
It is an interesting fact that every ~~party of 6 people *must*~~ contain either three mutual acquaintances or three mutual non-acquaintances.

Start with guest A.

Among the remaining 5 guests, A has either at least three acquaintances or at least three non-acquaintances.

By PHP strong form

Case 1: Suppose A has three acquaintances: B, C, D.

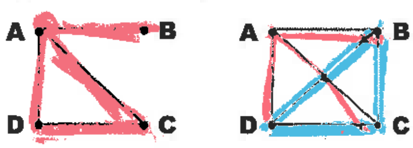


If any two of these are acquainted, we have three mutual acquaintances. WLOG D, C

If no two of these are acquainted, we have three mutual non-acquaintances!

$R(3,3) \le 6$

Case 2: Suppose A has three non-acquaintances: B, C, D.



If any two of these are non-acquainted, we have three mutual non-acquaintances. WLOG DC

If no two of these are non-acquainted, we have three mutual acquaintances!

$\Rightarrow R(3,3) = 6$

The *Ramsey Number* $R(m, n)$ gives the minimum number of people at a party that will guarantee the existence of either m mutual acquaintances or n mutual non-acquaintances.

blue K_m

red K_n

We just constructed a proof that $R(3, 3) = 6$.

Ramsey's Theorem guarantees that $R(m, n)$ exists for any m and n.

Intriguingly, there is still no known procedure for *finding* Ramsey numbers!

It has actually been known since 1955 that $R(4, 4) = 18$.

We do not know $R(5, 5)$, but we do know that it lies somewhere between 43 and 49.

All we really know about $R(6, 6)$ is that it lies somewhere between 102 and 165.

There is a cash prize for finding either one.

The great mathematician Paul Erdös was fascinated by the difficulty of finding Ramsey numbers. Here's what he had to say:

"Imagine an alien force vastly more powerful than us landing on Earth and demanding the value of $R(5, 5)$ or they will destroy our planet.

In that case, we should marshal all our computers and all our mathematicians and attempt to find the value.

But suppose, instead, that they ask for $R(6, 6)$.

Then we should attempt to destroy the aliens."

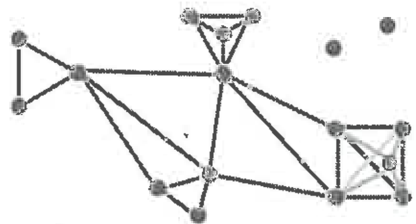
We can all do math every day!

NCTM

http://www.cbs.com/primetime/numb3rs/ti/activities.shtml

FYI

Creating a simplicial complex (a type of hypergraph)

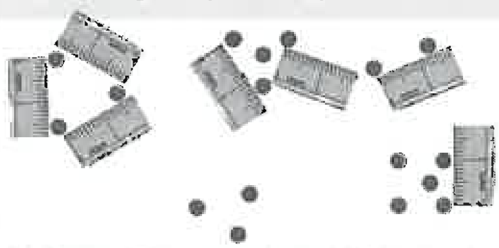


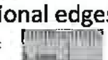
only vertices & edges ⇒ graph

Creating a simplicial complex

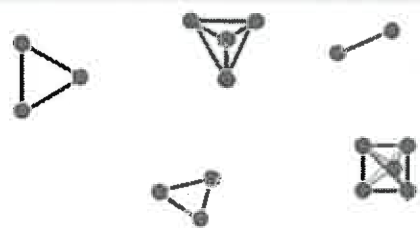
3.) Add 2-dimensional triangles
and 3-dimensional tetrahedrons (3-simplices)
and higher dimensional simplices if you so choose.

Creating a simplicial complex from data.



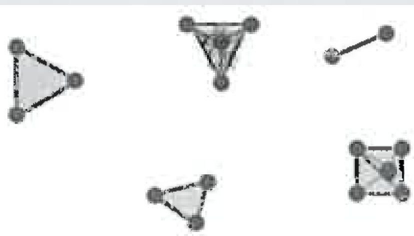
1.) Adding 1-dimensional edges (1-simplices)
Let T = Threshold = 
Connect vertices v and w with an edge iff
the distance between v and w is less than T

Creating the Vietoris Rips simplicial complex



1.) Adding 1-dimensional edges (1-simplices)
Add an edge between data points that are "close"

Creating the Vietoris Rips simplicial complex



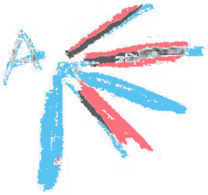
2.) Add 2-dimensional triangles (2-simplices)
Add all possible 2-simplices.

{a, b, c, d}

Creating the Vietoris Rips simplicial complex

{u, v, w, x, y}

2.) Add all possible simplices of dimensional > 1.



$R_A \leftarrow$ $B_A \leftarrow$ $r(s,t)$

Thm (Erdos and Szekeres): $r(m,n)$ is finite for all $s,t \geq 2$. If $m > 2$, $n > 2$, then

$$r(m,n) \leq r(m-1,n) + r(m,n-1)$$

$$r(m,n) \leq \binom{m+n-2}{m-1}$$

Thm: $r(m,n) \leq r(m-1,n) + r(m,n-1)$.

Proof: Let $p = r(m-1,n) + r(m,n-1)$. ← an arbitary coloring

Color the edges of K_p red and blue.

Claim: There exists a red K_m or a blue K_n.

Let A be a vertex of K_p.

Let $R_A = \{v \mid \{v,A\} \text{ is red }\}$. Let $B_A = \{v \mid \{v,A\} \text{ is blue }\}$.

Note that $|R_A| + |B_A| = p - 1 = r(m-1,n) + r(m,n-1) - 1$.

Thus by the PHP (strong form),
either $|R_A| \geq r(m-1,n)$ OR $|R_B| \geq r(m,n-1)$.

Case 1: $|R_A| \geq r(m-1,n)$. red Blue

Let $q = |R_A| \geq r(m-1,n)$.

Let K_q be the complete subgraph of K_p containing the q vertices in R_A.

$q \geq r(m-1,n)$ implies

Case 2: $|R_B| \geq r(m,n-1)$: Similar to case 1.

$r(s_1, ..., s_k) = min\{n \mid$ if the edges of K_n are colored using k colors, there exist an i colored $K_{s_i}\}$ ← K_{s_i} colored w/ color i

Ex: $r(s,2,2) = min\{n \mid$ if the edges of K_n are colored using red, white, and blue, then $\exists$ a red K_s or a white K_2 or a blue $K_2\}$.

Claim: $r(s,2,2) = s$.

Proof: If all the edges of K_s are colored red, then $\exists$ red K_s. If not, then $\exists$ an edge colored either white or blue. Thus we have a white K_2 or a blue K_2.

Hypergraph: (V,E), $E \subset \mathcal{P}(V)$ = power set of V = the set of all subsets of V.

Ex: $P(\{a,b\}) = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$. FYI

$X^{(t)}$ = set of all t-tuples of X.

Ex: If $X = \{a,b,c,d\}$ then ← graph

$X^{(2)} = \{\{a,b\},\{a,c\},\{a,d\},\{b,c\},\{b,d\},\{c,d\}\} = K_4$

$X^{(3)} = \{\{a,b,c\},\{a,b,d\},\{a,c,d\},\{b,c,d\}\}$

$X^{(4)} = \{\{a,b,c,d\}\}$

A coloring of edges: $c : X^{(t)} \to \{red,\ blue\}$

$Y^{(t)} \subset X^{(t)}$ is a red n set if $|Y^{(t)}| = n$ and $c(Y^{(t)}) = $ red.

$r_t(n_1,n_2) = min\{m \mid |X| = m$ implies $X^{(t)}$ has a red n_1 set or a blue n_2 set $\}$

$r_2(s,t) = r(s,t)$