Math 150 Exam 1 or see also exam 1 from 2006
September 10, 2009

Choose 6 from the following 8 problems. Circle your choices: 1 2 3 4 5 6 7 8
You may do more than 6 problems in which case one of your two unchosen problems can replace your lowest problem at ~~2/3~~ 4/5 the value (or more) as discussed in class.

1.) Determine which of the following sequences are inversion sequences. For each inversion sequence, determine its corresponding permutation. State whether the permutation is even or odd. If the sequence is not an inversion sequence, state why you know it is not.

0321401

2103000

51023020

0243100

2a.) Determine the inversion sequence for 526314

2b.) Which permutation of $\{1, 2, 3, 4, 5, 6\}$ follows 526314 in using the algorithm described in Section 4.1? Explain.

HW 2.

38) The number of integral solutions to $x_1 + x_2 + x_3 + x_4 = 30$ where $x_1 \geq 2$, $x_2 \geq 0$, $x_3 \geq -5$, $x_4 \geq 8$ is the same number of solutions as $(y_1 + 2) + y_2 + (y_3 - 5) + (y_4 + 8) = 30$ where $y_1 = x_1 - 2 \geq 2 - 2$, $y_2 = x_2 \geq 0$, $y_3 = x_3 + 5 \geq -5 + 5$, $y_4 = x_4 - 8 \geq 8 - 8$ which has the same number of solutions as $y_1 + y_2 + y_3 + y_4 = 25$ where $y_1 \geq 0$, $y_2 \geq 0$, $y_3 \geq 0$, $y_4 \geq 0$.

Hence the number of integral solutions to $x_1 + x_2 + x_3 + x_4 = 30$ where $x_1 \geq 2$, $x_2 \geq 0$, $x_3 \geq -5$, $x_4 \geq 8 = \binom{25+4-1}{25} = \binom{28}{25}$

39a) C(20, 6)

b.) Similar to 11 (but choosing 6 instead of 3)

Suppose the i_1th, i_2th, i_3th, i_4th, i_5th, i_6th sticks were removed.

Let $x_1 = i_1 - 1 =$ number of sticks before the i_1th stick.

Let $x_j = i_j - i_{j-1} - 1$, for $j = i_2, ..., i_6 =$ number of sticks between the jth and (j + 1)th sticks.

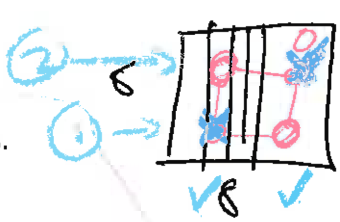
Let $x_7 = 20 - i_6 =$the number of sticks after the i_6th stick.

The number of ways of removing 6 sticks from 20 such that no two are consecutive is the same as the number of integral solutions to $x_1+x_2+x_3+x_4+x_5+x_6+x_7 = 14$ where $x_1, x_7 \geq 0$ and $x_i \geq 1$ for $i = 2, 3, 4, 5, 6$. This is the same as the number of solutions to $x_1+y_2+1+y_3+1+y_4+1+y_5+1+y_6+1+x_7 = 14$ where $x_1, x_7 \geq 0$, $y_2 = x_2 - 1 \geq 1 - 1 = 0$, $y_i = x_i - 1 \geq 1 - 1 = 0$ for $i = 2, 3, 4, 5, 6$. This is the same as the number of solutions to $x_1 + y_2 + y_3 + y_4 + y_5 + y_6 + x_7 = 9$ where $x_1, x_7, y_i \geq 0$ for $i = 2, 3, 4, 5, 6$.

Hence by thm 3.5.1, the answer is $\binom{9+7-1}{9} = \binom{15}{9}$

c.) In this case we need to know the number of solutions to $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 14$ where $x_1, x_7 \geq 0$ and $x_i \geq 2$ for $i = 2, 3, 4, 5, 6$. This is the same as the number of solutions to $x_1 + y_2 + 2 + y_3 + 2 + y_4 + 2 + y_5 + 2 + y_6 + 2 + x_7 = 14$ where $x_1, x_7 \geq 0$, $y_2 = x_2 - 1 \geq 1 - 1 = 0$, $y_i = x_i - 2 \geq 2 - 2 = 0$ for $i = 2, 3, 4, 5, 6$. This is the same as the number of solutions to $x_1 + y_2 + y_3 + y_4 + y_5 + y_6 + x_7 = 4$ where $x_1, x_7, y_i \geq 0$ for $i = 2, 3, 4, 5, 6$.

Hence by thm 3.5.1, the answer is $\binom{4+7-1}{4} = \binom{10}{4}$



5 rooks

50) Place two of the rooks in two non attacking positions.

Choose two of 8 columns: $C(8, 2) = \frac{(8!)}{(6!)(2!)}$

C(8,2)

Note when choosing the two columns, the order does NOT matter.

Place one rook in these 2 columns in two different rows: $P(8, 2) = \frac{(8!)}{(6!)}$ /2

Note when choosing the two rows, the order DOES matter. For example, suppose columns a and b were first chosen. If rows c and d are chosen in the order c, d, then the rooks are placed in squares $(a, c), (b, d)$ while if the rows c and d are chosen in the order d, c, then the rooks are placed in squares $(a, d), (b, c)$

Place two more rooks to form a rectangle: 1 choice

But to define rectangle choosing order of rows doesn't matter

$$\left[C(8,2)\cdot C(8,2)\right](1)(64-4)$$

Place the fifth rook: 64 - 4 = 60 choices

Place two of the rooks in two non attacking positions AND place two more rooks to form a rectangle AND place the fifth rook: $\frac{(8!)}{(6!)(2!)}\frac{(8!)}{(6!)}\frac{(60)}{2}$

51) Number of n-combinations from the multiset $\{n\cdot a, 1, 2, ..., n\}$

Choose k elements from $S = \{1, 2, ..., n\}$: $C(n, k)$

Choose $n-k$ elements from $T = \{n \cdot a\}$: 1

Choose k elements from $\{1, 2, ..., n\}$ AND choose $n-k$ elements from $\{n \cdot a\}$: $C(n, k)$

Choose (0 elements from S AND n elements from T) OR (1 element from S AND $n-1$ elements from T) OR (2 elements from S AND $n-2$ elements from T) OR ... OR (n elements from S AND 0 elements from T):

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + ... + \binom{n}{n} = 2^n$$

52) Number of n-combinations from the multiset $\{n \cdot a, n \cdot b, 1, 2, ..., n+1\}$

Choose k elements from $S = \{1, 2, ..., n+1\}$: $C(n+1, k)$

Choose a number j such that $0 \leq j \leq n-k$: $n-k+1$:

Choose j elements from T and $n-k-j$ elements from R: 1

Choose k elements from S AND choose $n-k$ elements from $T \cup R$: $(n-k+1)C(n+1, k) = (n+1)C(n, k)$

Choose (0 elements from S AND n elements from $T \cup R$) OR (1 element from S AND $n-1$ elements from $T \cup R$) OR (2 elements from S AND $n-2$ elements from $T \cup R$) OR ... OR (n elements from S AND 0 elements from $T \cup R$):

$$(n+1)[\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + ... + \binom{n}{n}] = (n+1)2^n$$

If $p \Rightarrow q$ is true, then
its contrapostive $\sim q \Rightarrow \sim p$ is also true.

But its converse, $q \Rightarrow p$ may not be true.

Thm 2.1.1: Pigeonhole Principle (weak form): If you have $n+1$ objects placed in n boxes, then at least one box will be occupied by 2 or more objects.

Thm 2.1.1: Pigeonhole Principle (weak form): If you have $n+1$ pigeons in n pigeonholes, then at least one pigeonhole will be occupied by 2 or more pigeons.

Thm 2.1.1: If $f : A \to B$ is a function and $|A| = n + 1$, and $|B| = n$, then f is not 1:1.

Cor: If $f : A \to B$ is a function and A is finite and $|A| > |B|$, then f is not 1:1.

Note that the domain must have more elements then the codomain to **guarantee** that f is not 1:1.

Recall the *converse* of [p implies q] is [q implies p].

Note the converse of a theorem is frequently false as the following example illustrates:

$$c : \{1, ..., n\} \to \{1, ..., n\},\ c(k) = 1 \text{ is not } 1:1,$$

but domain does not have more elements than the codomain.

$f : A \to B$ a function which is not 1:1 does not imply $|A| > |B|$.

Contrapositive of [p implies q] is [$\sim q$ implies $\sim p$].
The contrapositive of a theorem is true:

Cor: If $f : A \to B$ is a function which is 1:1, then $|A| \leq |B|$.

Related theorem:
Thm: If $f : A \to B$ is a function and
if $|A| = n = |B|$, then f is 1:1 iff f is onto.

$n \quad (n-1) \quad \_ \qquad (1)$

$r = n$

n-Permutation of n objects: $P(n,n) = n! =$ number of ways to place n nonattacking rooks on an $n \times n$ chessboard.

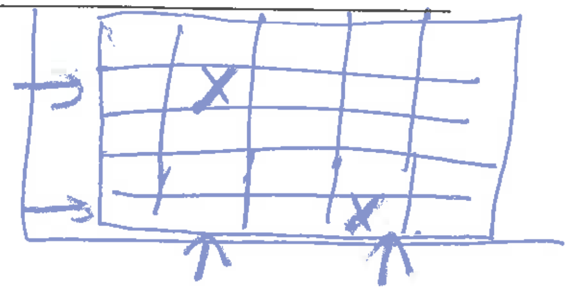
$r \leq n$

r-Permutation of n objects: $P(n,r) = n(n-1)...(n-r+1) =$ number of ways to place r nonattacking rooks on an $r \times n$ chessboard, $r \leq n$. $\quad P(n,r) = \frac{n!}{r!}$

$C(n,r)P(n,r) = \frac{[n(n-1)...(n-r+1)]^2}{r!} =$ number of ways to place r nonattacking rooks on an $n \times n$ chessboard, $r \leq n$.

Permut

2.4 Permutations of Multisets



Thm 2.4.1: Let $A = \{\infty \cdot 1, \infty \cdot 2, ..., \infty \cdot k\}$

The number of r permutations of $A = k^r$.

Thm 2.4.2: Let $B = \{n_1 \cdot 1, n_2 \cdot 2, ..., n_k \cdot k\}$

$n = n_1 + n_2 + ... + n_k$

The number of n-permutations of $B = \frac{n!}{n_1!n_2!\cdots n_k!}$.

If want r-permutations of B, need to use "or" statements or technique from later chapter.

Combinations: order does NOT matter

$\binom{n}{r} =$ # of r combinations of $\{1, 2, ..., n\}$ $= \frac{n!}{(n-r)!\,r!}$

$=$ subsets of $\{1, 2, ..., n\}$ containing exactly r elements.

$2^n = \Sigma_{i=0}^{n} \binom{n}{i} =$ # of subsets of $\{1, 2, ..., n\}$.

for simplification purposes and for comb proof

Pascal's Triangle: $C(n,r) = C(n-1, r-1) + C(n-1, r)$

combinations

2.4 Combinations of Multisets

Let $A = \{\infty \cdot 1, \infty \cdot 2, ..., \infty \cdot k\}$ or equiv $A = \{r \cdot 1, r \cdot 2, ... r \cdot k\}$

The number of r combinations of $A = \binom{r+k-1}{r}$

$=$ # of solutions to $x_1 + x_2 + ... + x_k = r$ such that $x_i \geq 0, x_i \in \mathcal{Z}$

$=$ # of permutations of $\{r \cdot 1, (k-1) \cdot +\}$

$=$ partitions of r indistinguishable objects into k distinguishable boxes.

if need $x_i \geq a_i$ replace x_i with $y_i + a_i$ since $y_i = x_i - a_i \geq 0$

$r+1$ combination $\Rightarrow$ or statements