$\{n_1 \cdot a_1, n_2 \cdot a_2, \ldots n_k \cdot a_k\}$ where $\boxed{\sum_{i=1}^{k} n_i = r}$ ← r elements in multi-subset

## 2.5 Combinations of Multisets

Thm 2.5.1 Let $S = \{\infty \cdot a_1, ..., \infty \cdot a_k\}$. Then the number of $r$-combinations of $S$ is $\frac{(r+k-1)!}{r!\,(k-1)!}$.

Proof: The number of $r$-combinations of $S$ ← order does not matter

= the number of integral solutions to the equation

$$x_1 + x_2 + ... + x_k = r \qquad (*)$$

where $x_i \geq 0\ \forall i$ (and where $x_i$ = the number of $a_i$'s chosen for an $r$−combination). order of $x_i$'s matter

= the number of permutations of $\{r \cdot 1, (k-1) \cdot +\}$ by the following: (order matters

Suppose $(c_1, c_2, ..., c_k)$ is a solution to (*). This corresponds to the permutation $11...1 + 1..1 + ... + 11..1$,

where there are $k-1$ +'s and $c_1$ 1's before the first +, $c_i$ 1's between the (i-1)th and ith +'s for $i = 2, ..., k-1$, and $c_k$ 1's after the last +. Since $c_1 + c_2 + ... + c_k = r$, there are $r$ 1's, and thus $11...1 + 1..1 + ... + 11..1$ is a permutations of $\{r \cdot 1, (k-1) \cdot +\}$.

A permutation of $\{r \cdot 1, (k-1) \cdot +\}$ corresponds to a solution $(c_1, c_2, ..., c_k)$ of (*) where $c_1$ = the number of 1's before the first +, $c_i$ = the number of 1's between the (i-1)th and ith +'s for $i = 2, ..., k-1$, and $c_k$ = the number of 1's after the last +. Since there are $r$ 1's, $c_1 + c_2 + ... + c_k = r$.

The number of permutations of $\{r \cdot 1, (k-1) \cdot +\}$ is

$$= \boxed{\frac{(r+k-1)!}{r!\,(k-1)!}}$$

Corollary: Let $S = \{r \cdot a_1, ..., r \cdot a_k\}$. Then the number of $r$-combinations of $S$ is → $\frac{(r+k-1)!}{r!\,(k-1)!}$

Proof: The # of $r$ combinations of $\{r \cdot a_1, \ldots r \cdot a_k\}$
$= \#$ of $r$ comb of $\{\infty \cdot a_1, \ldots, \infty \cdot a_k\}$
An $r$ comb in $\{r \cdot a_1, \ldots r \cdot a_k\}$ is also an $r$ comb in $\{\infty \cdot a_1 \ldots \infty \cdot a_k\}$ and vice versa

Some examples

$S = \{\infty \cdot a_1, \infty \cdot a_2, ..., \infty \cdot a_5\}$.

$\{0 \cdot a_1, 0 \cdot a_2, 3 \cdot a_3, 0 \cdot a_4, 1 \cdot a_5\}$

Then a 4-combination of $S$ is $\{a_3, a_3, a_3, a_5\}$

Suppose $x_1 + x_2 + x_3 + x_4 + x_5 = \boxed{4}$. ← $0+0+3+0+1 = 4$
Then $(x_1, x_2, x_3, x_4, x_5) = (0, 0, 3, 0, 1)$ is a solution.

$+ + 111 + +1$ is a permutation of $\{4 \cdot 1, (5-1) \cdot +\}$

$(x_1, x_2, x_3, x_4, x_5) = (2, 1, 0, 1, 0)$ is a solution to $x_1 + x_2 + x_3 + x_4 + x_5 = 4$.

$2+1+0+1+0$

$11 + 1 + +1+$ is a permutation of $\{4 \cdot 1, (5-1) \cdot +\}$

A 4-combination of $S$ is $\{a_1, a_1, a_2, a_4\}$

~~$++++4$~~ is a permutation of $\{4 \cdot 1, (5-1) \cdot +\}$

A 4-combination of $S$ is $\{a_5, a_5, a_5, a_5\}$

$(x_1, x_2, x_3, x_4, x_5) = (0, 0, 0, 0, 4)$ is a solution to $x_1 + x_2 + x_3 + x_4 + x_5 = 4$.

$+ + + + 1111$

$0 + 0 + 0 + 0 + 4$

$x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = 0 + x_5 = 4$