Math 150 Exam 1
September 10, 2009

1.) Determine which of the following sequences are inversion sequences. For each inversion sequence, determine its corresponding permutation. State whether the permutation is even or odd. If the sequence is not an inversion sequence, state why you know it is not.

0321401: Not an inversion sequence. The last term is 1, but this term must be 0 since there aren't any numbers larger than the largest number in a permutation.

2103000: permutation = 3215674

2 + 1 + 3 = 6. Thus permutation is even.

51023020: Not an inversion sequence. The second to the last term is 2, but this term must be either 0 or 1 since there is only one number larger than the second to the largest number in a permutation.

0243100: 1652743

2 + 4 + 3 + 1 = 10. Thus permutation is even.

2a.) Determine the inversion sequence for 526314

412200

2b.) Which permutation of $\{1, 2, 3, 4, 5, 6\}$ follows 526314 in using the algorithm described in Section 4.1? Explain.

Remove largest term from 526314: 52314. Its inversion sequence is 31110. 3 + 1 + 1 + 1 = 6. Thus 52314 is an even permutation. Hence 6 is moving left. Thus 562314 follows 526314.

3a.) Determine the number of linear permutations of the multiset $\{1 \cdot a,\ 5 \cdot b,\ 10 \cdot c,\ 20 \cdot d\}$.

$\frac{36!}{5!10!20!}$

3b.) Determine the number of circular permutations of the multiset $\{1 \cdot a,\ 5 \cdot b,\ 10 \cdot c,\ 20 \cdot d\}$.

Place a first. The number of ways to place the remaining letters is

$\frac{35!}{5!10!20!}$

3c.) Determine the number of linear 5-permutations of the set $\{1, 2, ..., 20\}$. $P(20, 5) = \frac{20!}{15!}$

3d.) Determine the number of circular 5-permutations of the set $\{1, 2, ..., 20\}$.

$P(20, 5)/5 = \frac{20!}{(15!)(5)}$

4a.) In how many ways can 16 blue rooks be placed on a 30-by-30 chessboard in non-attacking position?

Choose 6 columns: $C(30,16)$

Choose where to place the rooks in these 6 columns: $P(30,16)$

$C(30,16)P(30,16) = \frac{30!}{16!14!}\frac{30!}{14!}$

4b.) In how many ways can 16 blue rooks be placed on a 30-by-30 chessboard in non-attacking position so that the first column is NOT empty?

Choose 16 columns. Note first column must be chosen, so we only need to choose 15 additional columns. There are $C(29,15)$ ways to do this.

Now choose where to place the rooks in these 16 columns: $P(30,16)$

$C(29,15)P(30,16) = \frac{29!}{15!14!}\frac{30!}{14!}$

4c.) In how many ways can 10 blue rooks, 5 green rooks, and 1 red rook be placed on a 30-by-30 chessboard in non-attacking position so that the first column is NOT empty?

Number of ways to place rooks = answer from 4b = $\frac{29!}{15!14!}\frac{30!}{16!}$

Number of ways to color these rooks: $\frac{16!}{10!5!}$

Thus answer to 4c is $[\frac{29!}{15!14!}\frac{30!}{14!}][\frac{16!}{10!5!}]$

5a.) How many sets of 4 numbers can be formed from the numbers $\{1,2,...,310\}$ if no two consecutive numbers are to be in a set?

Let x_1 = the number of numbers before the first chosen number. Note $x_1 \geq 0$.

Let x_i = the number of numbers between the ith and $(i+1)th$ chosen number for $i = 2,3,4$. Since the numbers are not consecutive, $x_i > 0$ for $i = 2,3,4$.

Let x_5 = the number of numbers after the last chosen number. Note $x_5 \geq 0$.

Since 4 numbers have been chosen, $x_1 + x_2 + x_3 + x_4 + x_5 = 310 - 4$

The number of sets of 4 numbers that can be formed from the numbers $\{1,2,...,310\}$ if no two consecutive numbers are to be in a set = the number of integral solutions to $x_1 + x_2 + x_3 + x_4 + x_5 = 306$ where $x_1 \geq 0$, $x_i \geq 1$ for $i = 2,3,4$, $x_5 \geq 0$.

Let $y_i = x_i - 1$ for $i = 2,3,4$. Then $y_i \geq 0$. $\quad x_1 + y_2 + y_3 + y_4 + x_5 = 306 - 3 = 303$

The number of nonnegative solutions to this equation is $\frac{(303+5-1)!}{303!4!} = \frac{307!}{303!4!}$

Hence answer is $\frac{307!}{303!4!}$.

5b.) There are 310 identical sticks lined up in a row occupying 310 distinct places. 4 of the sticks are to be chosen. How many choices are there if no two of the chosen sticks can be consecutive?

$\frac{307!}{303!4!}$. Note this is exactly the same question as in 5a, but worded differently.

6.) Prove that the number of nonnegative integral solutions to $x_1 + x_2 = 8$ is the same as the number of permutations of the multiset $\{8 \cdot 1, +\}$

Define f : nonnegative integer solutions of $x_1 + x_2 = 8 \to$ permutations of the multiset $\{8 \cdot 1, +\}$ by

$f(c_1, c_2) =$ the permutation $1, 1, ..., 1, +, 1, 1, ..., 1$ where there are c_1 1's before the $+$ and c_2 1's after the $+$. Since (c_1, c_2) is a solution to $x_1 + x_2 = 8$, $f(c_1, c_2)$ is a permutation of the multiset $\{8 \cdot 1, +\}$. Thus f is well-defined.

Define g : permutations of the multiset $\{8 \cdot 1, +\} \to$ nonnegative integer solutions of $x_1 + x_2 = 8$ by

$g(1, 1, ..., 1, +, 1, 1, ..., 1) = (c_1, c_2)$ where $c_1 =$ the number of 1's before the $+$ and $c_2 =$ the number of 1's after the $+$. Since there are eight 1's, (c_1, c_2) is a solution to $x_1 + x_2 = 8$. Thus g is well-defined.

Note g is the inverse of f. Hence f is a bijection.

7.) $r(s, 2) = \underline{s}$. Prove this **equality**.

$r(s, 2) = min\{n \mid$ if the edges of K_n are colored red and blue, then there exists either a red K_s or a blue $K_2\}$

Claim 1: $r(s, 2) \geq s$

Proof of Claim 1: Color all the edges of K_{s-1} red. Then there does not exist a red K_s or a blue K_2 in this coloring of K_{s-1}. Thus $r(s, 2) > s - 1$

Claim 2: $r(s, 2) \leq s$

Proof of Claim 2: Color the edges of K_s using the two colors, red and blue.

Case 1: All the edges are red. In this case, there exists a red K_s.

Case 2: Not all the edges are red. Thus this coloring has a blue edge. In this case, there exists a blue K_s.

Thus $r(s, 2) \leq s$

By Claim 1 and 2, $r(s, 2) = s$

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8) Show that given m integers $a_1, a_2, ..., a_m$, there exists k and l with $0 \leq k < l \leq m$ such that $a_{k+1} + a_{k+2} + ... + a_l$ is divisible by m. Ans: See Application 3, p. 71.