7.4: linear homogeneous recurrence relation:

Defn: A recurrence relation is *linear* if

$$h_n = a_1(n)h_{n-1} + a_2(n)h_{n-2} + ... + a_k(n)h_{n-k} + b(n)$$

A recurrence relation has order k if $a_k \neq 0$

Ex: Derangement

$$D_n = (n-1)D_{n-1} + (n-1)D_{n-2}, \qquad D_1 = 0, \quad D_2 = 1$$

$$D_n = nD_{n-1} + (-1)^n, \qquad D_1 = 0$$

Fibonacci: $f_n = f_{n-1} + f_{n-2}, \qquad f(0) = 0, \ f(1) = 1$

Defn: A linear recurrence relation is *homogeneous* if $b = 0$

Defn: A linear recurrence relation has *constant coefficients* if the a_i's are constant.

Tentative HW 12: Ch 14: 1, 4, 5, 10, 13, 18, 22, 24, 25 and

A.) Suppose the sequences r_n, s_n, and t_n satisfy the homogeneous linear recurrence relation,

$$h_n = a_1(n)h_{n-1} + a_2(n)h_{n-2} + a_3(n)h_{n-3} \ (**).$$

Show that the sequence, $c_1r_n + c_2s_n + c_3t_n$ also satisfies this homogeneous linear recurrence relation (**).

B.) Suppose the sequence ψ_n satisfies the linear recurrence reln, $h_n = a_1(n)h_{n-1} + a_2(n)h_{n-2} + a_3(n)h_{n-3} + b(n)$ (*). Show that the sequence, $c_1r_n + c_2s_n + c_3t_n + \psi_n$ also satisfies this linear recurrence relation.

C.) How many terms of the sequence are needed to find a unique sequence with these terms satisfying (*). What linear system of equations can be used to determine c_1, c_2, c_3.

7.4: linear homogeneous recurrence relation w/constant coefficients:

Ex: Solve the recurrence relation: $h_n + h_{n-2} = 0$, $h_0 = 3, h_1 = 5$

Guess q^n is a solution.

$q^n + q^{n-2} = q^{n-2}(q^2 + 1) = 0$ $\qquad$ $q^2 + 1 = 0$ implies $q = \pm i$

Thus the general solution is $h_n = c_1 i^n + c_2(-i)^n$

i.e., this function satisfies the recurrence relation.

Now need to find c_i's resulting in initial conditions:

$h_0 = 3$: $c_1 + c_2 = 3$ implies $c_2 = 3 - c_1$

$h_1 = 5$: $c_1 i - c_2 i = 5$ implies $-c_1 + c_2 = 5i$

$-c_1 + 3 - c_1 = 5i$. Thus $-2c_1 + 3 = 5i$

Hence $c_1 = \frac{3-5i}{2}$ and $c_2 = 3 - (\frac{3-5i}{2}) = \frac{3+5i}{2}$

$h_n = (\frac{3-5i}{2})i^n + (\frac{3+5i}{2})(-i)^n$ satisfies the recurrence relation and the initial conditions.

$h_n = i^n[(\frac{3-5i}{2}) + (\frac{3+5i}{2})(-1)^n] = i^n[(\frac{3}{2})(1 + (-1)^n) + (\frac{5i}{2})(-1 + (-1)^n)]$

$h_{2j} = (\frac{3-5i}{2})i^{2j} + (\frac{3+5i}{2})(-i)^{2j} = 3(-1)^j$

$h_{2j+1} = (\frac{3-5i}{2})i^{2j+1} + (\frac{3+5i}{2})(-i)^{2j+1} = -5(i)^{2j+2} = 5(-1)^j$

Thus starting with h_0, we have the sequence:

3, 5, -3, -5, 3, 5, -3, -5, 3, 5, -3, -5, 3, 5, ...

Ex: Solve the recurrence relation, $h_n - 2h_{n-1} + 2h_{n-3} - h_{n-4} = 0$, $h_0 = 3, h_1 = 3, h_2 = 7, h_3 = 15$.

Guess q^n is a solution.

$q^n - 2q^{n-1} + 2q^{n-3} - q^{n-4} = q^{n-4}(q^4 - 2q^3 + 2q - 1) = 0$,

$q^{n-4}(q^3 - 3q^2 + 3q - 1)(q + 1) = q^{n-4}(q - 1)^3(q + 1) = 0$

q = 1 , 1, 1, -1

Note: 1 is a **repeated root**

Note $n^j(1)^n$, $j = 0, 1, 2$, are solutions to the recurrence relation.

Check: If $h_n = (1)^n = 1$: $1 - 2 + 2 - 1 = 0$.

Check: If $h_n = n(1)^n = n$:

$n - 2(n - 1) + 2(n - 3) - (n - 4) = n - 2n + 2n - n + 2 - 6 + 4 = 0$

Check: If $h_n = n^2(1)^n = n^2$:

$n^2 - 2(n - 1)^2 + 2(n - 3)^2 - (n - 4)^2 =$
$n^2 - 2(n^2 - 2n + 1) + 2(n^2 - 6n + 9) - (n^2 - 8n + 16) = 0$

General solution

$h_n = c_1(1)^n + c_2n(1)^n + c_3n^2(1)^n + c_4(-1)^n = c_1 + c_2n + c_3n^2 + c_4(-1)^n$

Now need to find c_i's resulting in initial conditions:

$h_0 = 3 = c_1 + c_4$

$h_1 = 3 = c_1 + c_2 + c_3 - c_4$

$h_2 = 7 = c_1 + 2c_2 + 4c_3 + c_4$

$h_3 = 15 = c_1 + 3c_2 + 9c_3 - c_4$

$h_0 = 3 = c_1 + c_4$

$h_1 = 3 = c_1 + c_2 + c_3 - c_4$

$h_2 = 7 = c_1 + 2c_2 + 4c_3 + c_4$

$h_3 = 15 = c_1 + 3c_2 + 9c_3 - c_4$

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 3 \\ 1 & 1 & 1 & -1 & 3 \\ 1 & 2 & 4 & 1 & 7 \\ 1 & 3 & 9 & -1 & 15 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 2 & 4 & 0 & 4 \\ 0 & 3 & 9 & -2 & 12 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 2 & 4 & 4 \\ 0 & 0 & 6 & 4 & 12 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 2 & 4 & 4 \\ 0 & 0 & 0 & -8 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 2 & 4 & 4 \\ 0 & 0 & 0 & -8 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Thus $c_1 = 3$, $c_2 = -2$, $c_3 = 2$, $c_4 = 0$.

$h_n = c_1 + c_2 n + c_3 n^2 + c_4(-1)^n = 3 - 2n + 2n^2$

$$\textbf{Hence } h_n = 3 - 2n + 2n^2$$

Check Initial Conditions: $h_0 = 3, h_1 = 3, h_2 = 7, h_3 = 15$

$h_0 = 3 - 0 + 0 = 3$ $\qquad$ $h_1 = 3 - 2 + 2 = 3,$

$h_2 = 3 - 4 + 8 = 7$ $\qquad$ $h_3 = 3 - 6 + 18 = 15.$

7.4: linear homogeneous recurrence relation:

$$h_n - a_1 h_{n-1} - a_2 h_{n-2} - \ldots - a_k h_{n-k} = 0$$

Suppose $\phi(n)$ and $\psi(n)$ are solns to the above recurrence relation, then

Claim 1: $c\phi(n)$ is a solution for any constant c

Claim 2: $\phi(n) + \psi(n)$ is also a solution.

Hence if $\phi_i(n)$ are solns, then $\Sigma c_i \phi_i(n)$ is a soln for any constants c_i.

Thm 7.4.1: Suppose a_i are constants and $q \neq 0$. Then q^n is a solution to

$$h_n - a_1 h_{n-1} - a_2 h_{n-2} - ... - a_k h_{n-k} = 0$$

iff q is a root of the polynomial equation

$$x^k - a_1 x^{k-1} - a_2 x^{k-2} - ... - a_k = 0$$

If this characteristic equation has k distinct roots, $q_1, q_2, ..., q_k$,

then $h_n = c_1 q_1^n + c_2 q_2^n + ... + c_k q_k^n$ is the general solution.

I.e, given any initial values for $h_0, h_1, ..., h_{k-1}$, there exists $c_1, c_2, ..., c_k$ such that $h_n = c_1 q_1^n + c_2 q_2^n + ... + c_k q_k^n$ satisfies the recurrence relation and the initial conditions.

Thm 7.4.2: Suppose q_i is an s_i-fold root of the characteristic equation. Then

$H_i(n) = c_1 q_i^n + c_2 n q_i^n + ... + c_{s_i} n^{s_i - 1} q_i^n$

is a solution to the recurrence relation.

If the characteristic equation has t distinct roots $q_1, ..., q_t$ with multiplicity $s_1, ..., s_t$, respectively, then

$h_n = H_1(n) + ...H_t(n)$ is a general solution.

7.5: Non-homogeneous Recurrence Relations.

$$h_n - a_1 h_{n-1} - a_2 h_{n-2} - ... - a_k h_{n-k} = b$$

Let $k(h) = h_n - a_1 h_{n-1} - a_2 h_{n-2} - ... - a_k h_{n-k}$

Suppose ϕ is a solution to the recurrence relation $k(h) = 0$
and β is a solution to the recurrence relation $k(h) = b$.

Claim: $\phi + \beta$ is a solution to

To solve a non-homogeneous recurrence relation.

Step 1: Solve homogeneous equation.

Recall if constant coeffficents, guess $h_n = q^n$ for homogeneous eq'n.

Step 2: Guess a solution to non-homogeneous equation,
by guessing a solution β_n similar to $b(n)$.

Step 3a: Note general solution is $\sum c_i \phi_i(n) + \beta(n)$.

Step 3b: Find c_i using initial conditions.

Ex: Solve the recurrence relation: $h_n + h_{n-2} = 14n$, $h_0 = 3, h_1 = 5$

Step 1: Guess q^n is a solution to homogeneous equation:

$$h_n + h_{n-2} = 0.$$

$q^n + q^{n-2} = q^{n-2}(q^2 + 1) = 0$ $\qquad$ $q^2 + 1 = 0$ implies $q = \pm i$

Thus the general solution to homogeneous equation is

$$h_n = c_1 i^n + c_2 (-i)^n$$

Step 2: Guess a solution to non-homogeneous equation:

$$h_n + h_{n-2} = 14n$$

Guess $\beta_n = xn + y$.

Plug β_n into non-homogeneous equation: $[xn+y]+[x(n-2)+y] = 14n$

Solve for x and y: $2xn + 2y - 2x = 14n$ implies $x = 7$ and $y = 7$.

Thus a solution to non-homogeneous equation is $\beta(n) = 7n + 7$.

Step 3a: Note general soln to non-homogeneous equation is

$$h_n = c_1 i^n + c_2(-i)^n + 7n + 7$$

Step 3b: Find c_i using initial conditions.

$$h_n + h_{n-2} = 14n,\ h_0 = 3, h_1 = 5$$

$h_0 = 3$: $c_1 i^0 + c_2(-i)^0 + 7(0) + 7 = 3$ implies $c_1 + c_2 = -4$

$h_1 = 5$: $c_1 i^1 + c_2(-i)^1 + 7(1) + 7 = 5$ implies $ic_1 - ic_2 = -9$

$$\begin{aligned} c_1 + c_2 &= -4 \\ -c_1 + c_2 &= -9i \end{aligned}$$

implies $c_1 = \frac{-4+9i}{2} = -2 + \frac{9i}{2}$ and $c_2 = \frac{-4-9i}{2} = -2 - \frac{9i}{2}$

$h_n = (-2 + \frac{9i}{2})i^n + (-2 - \frac{9i}{2})(-i)^n + 7n + 7$

$= (i^n)[(-2)(1 + (-1)^n) + (\frac{9i}{2})(1 - (-1)^n)] + 7n + 7$

$h_{2j} = (-1)^j(-4) + 7(2j) + 7 = 4(-1)^{j+1} + 7 + 14j$

$h_{2j+1} = (i^{2j+1})9i + 7(2j+1) + 7 = (i^{2j+2})9 + 14j + 14 = 9(-1)^{j+1} + 14j + 14$

Thus the sequence is 3, 5, 25, 37, 31, 33, 53, 65, 59, 61, 81, 93, ...